EXISTENCE AND UNIQUENESS OF A FIXED POINT THEOREM ON C_J- METRIC SPACES

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Abstract. In this paper, we generalize a $J-$ metric spaces, where it defined a metric space in three dimensions with a triangle inequality that includes a constant $b > 0$. We extend the notion of $J-$ metric spaces to C_J metric spaces that include a control function θ in three dimensions instead of the constant b.

1. Introduction

The fixed point theory is a new, essential theory, and its application is utilized in many fields, including Mathematics, Economics, and many others. For example, the impact of the fixed point theory in the fractional differential equations appear clearly to all the observers, see [3, 4]. The fixed point theory and the proof of the uniqueness were introduced by Banach [2], which was encouraging to all subsequent researchers to start working on this theory; see [5, 8].

These days, the fixed point is an active area wildly generalizing Banach, see [6, 7]. Generalization of the fixed point theory can be made in two ways, either a generalization of the Banach contraction to another linear or nonlinear contraction. The other way of extension is to generalize the metric

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spaces by either changing the triangle inequality, omitting the symmetry condition, or assuming that the self-distance is not necessarily zero.

Those generalizations are important due to the fact that more general spaces or contractions impact a greater number of applications that can be adapted to that results.

In this work, we introduce a new extension to $J-$ metric spaces, called C_J- metric spaces, where θ is the controlled function in the triangle inequality. We prove some fixed point results in this new type of metric space.

In the main result section, we prove the existence and the uniqueness of a fixed point for selfmappings on C_J- metric spaces, in Theorems 2.7 and 2.8, we consider self-mappings that satisfy linear contractions where in Theorem 2.9, we consider mappings that satisfy nonlinear contractions. Our finding generalizes many results in the literature.

We begin our preliminaries by recalling the definitions of J-metric spaces.

Definition 1.1. [1] Consider a nonempty set δ, and a function $J : \delta^3 \to [0, \infty)$. Let us define the set,

$$S(J, \delta, \phi) = \{\{\phi_n\} \subset \delta : lim_{n\to\infty} J(\phi, \phi, \phi_n) = 0\}$$

for all $\phi \in \delta$.

Definition 1.2. [1] Let δ be a set with at least one element and, $J : \delta^3 \to [0, \infty)$ that satisfies the mentioned below conditions:

(i) $J(\alpha, \beta, \gamma) = 0$ implies $\alpha = \beta = \gamma$ for any $\alpha, \beta, \gamma \in \delta$.

(ii) There are some $b > 0$, where for each $(\alpha, \beta, \gamma) \in \delta^3$ and $\{\nu_n\} \in S(J, \delta, \nu)$

$$J(\alpha, \beta, \gamma) \leq b\ limsup_{n\to\infty}(J(\alpha, \alpha, \nu_n) + J(\beta, \beta, \nu_n) + J(\gamma, \gamma, \nu_n)).$$

Then, (δ, J) is defined as a $J-$metric space. In addition, if $J(\alpha, \alpha, \beta) = J(\beta, \beta, \alpha)$ for each $\alpha, \beta \in \delta$, the pair (δ, J) is defined as a symmetric $J-$metric space.

2. MAIN RESULTS

In this paper, we will define C_J- metric spaces and prove the existence and the uniqueness of the fixed point of self-mapping.

Definition 2.1. Let δ is a non empty set and a function $C_J : \delta^3 \to [0, \infty)$. Then the set is defined as follows

$$S(C_J, \delta, \alpha) = \{\{(\alpha_n)\} \subset \delta : lim_{n\to\infty} C_J(\alpha, \alpha, \alpha_n) = 0\}$$

for each $\alpha \in \delta$.

Definition 2.2. Let be a set with at least one element and $C_J : \delta^3 \to [0, \infty)$ fulfill the following conditions:

(i) $C_J(\alpha, \beta, \gamma) = 0$ implies $\alpha = \beta = \gamma$ for any $\alpha, \beta, \gamma \in \delta$.

(ii) (ii) There exist a function $\theta : \delta^3 \to [0, \infty)$, where θ is a continuous function and

$$lim_{n\to\infty}\theta(\alpha, \alpha, \alpha_n)$$

is a finite and exist where,

$$C_J(\alpha, \beta, \gamma) \leq \theta(\alpha, \beta, \gamma) limsup_{n\to\infty}(C_J(\alpha, \alpha, \phi_n)+C_J(\beta, \beta, \phi_n)+C_J(\gamma, \gamma, \phi_n).$$

Then (δ, C_J) is defined as C_J-metric space. In addition, if

$$C_J(\alpha, \alpha, \beta) = C_J(\beta, \beta, \alpha)$$

for each $\alpha, \beta \in \delta$, then (δ, C_J) is defined as symmetric C_J-metric space.

Remark 2.3. Notice that, this symmetry hypothesis does not necessarily mean that

$$C_J(\alpha, \beta, \gamma) = C_J(\beta, \alpha, \gamma) = C_J(\gamma, \beta, \alpha) = \cdots .$$

We will start by presenting some properties in the topology of C_J-metric spaces.

Definition 2.4. (1) Let (δ, C_J) is a C_J- metric space. A sequence $\{\alpha_n\} \subset \delta$ is convergent to an element $\alpha \in \delta$ if $lim_{n\to\infty}\alpha_n = \alpha$,for $\{\alpha_n\} \in S(C_J, \delta, \alpha)$.

(2) Let (δ, C_J) is a C_J-metric space. A sequence $\{\alpha_n\} \subset \delta$ is called Cauchy iff $lim_{n,m\to\infty}C_J(\alpha_n, \alpha_n, \alpha_m) = 0$.
(3) A C_J- metric space is called complete if each Cauchy sequence in δ is convergent.
(4) In a C_J-metric space (α, C_J), if ψ is a continuous map at $\alpha_0 \in \delta$ then for each $\alpha_n \in S(C_J, \alpha, a_0)$ gives $\{\psi a_n\} \in S(C_J, \alpha, \psi a_0)$.

Proposition 2.5. *In a C_J-metric space (δ, C_J), if $\{\alpha_n\}$ converges, then it is convergent to one exact element in δ.*

Definition 2.6. Let (δ, C_{J_1}) and (Γ, C_{J_1}) are two C_J- metric spaces and $\psi : \delta \to \Gamma$ is a map . Then ψ is said to be a continuous at $a_0 \in \delta$ if, for each $\varepsilon > 0$, there is $\zeta > 0$ where, for each $a \in \delta$, $C_{J_2}(\psi a_0, \psi a_0, \psi\alpha) < \varepsilon$ whenever $C_{J_1}(a_0, a_0, \alpha) < \zeta$.

Theorem 2.7. *Let (δ, C_J) is a C_J- complete symmetric metric space, and $g : \delta \to \delta$ is a continuous map satisfies*

$$C_J(g\alpha, g\beta, g\gamma) \lneq P(C_J(\alpha, \beta, \gamma)) \ forall \ \alpha, \beta, \gamma \in \delta.$$

Where, $P : [0, +\infty) \to [0, +\infty)$ is a function and for all $t \in [0, +\infty)$,

$$t \succ x, P(t) \succ P(x).$$

$$lim_{n\to\infty}P^n(t) = 0 \ for \ each \ fixed \ t > 0.$$

Then, g has a unique fixed point in δ.

Theorem 2.8. *Let* (δ, C_J) *is a* C_J- *complete symmetric metric space and* $g : \delta \to \delta$ *be a mapping that satisfies,*

$$C_J(g\alpha, g\beta, g\gamma) \leq \phi(\alpha, \beta, \gamma) C_J(\alpha, \beta, \gamma), \ \forall \alpha, \beta, \gamma \in \delta,$$

where $\phi \in A$ *, and* $\phi : \delta^3 \to (0, 1)$*, such that*

$$\phi(g(\alpha, \beta, \gamma)) \leq \phi(\alpha, \beta, \gamma) \text{ and } \{g : \delta \to \delta\}$$

g *is a given mapping. Then* g *has a unique fixed point in* δ*.*

Theorem 2.9. *Let* (δ, C_J) *is a complete symmetric* C_J-*metric spaces,* $g : \delta \to \delta$ *is a continuous map where*

$$C_J(g\alpha, g\beta, g\gamma) \leq aC_J(\alpha, \beta, \gamma) + bC_J(\alpha, g\alpha, g\alpha) + cC_J(\beta, g\beta, g\beta) + dC_J(\gamma, g\gamma, g\gamma)$$

for each $\alpha, \beta, \gamma \in \delta$ *where*

$$0 < a + b < 1 - c - d,$$
$$0 < a < 1.$$

Then, there is a unique fixed point of g*.*

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