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FIXED POINT THEOREMS FOR θ-ϕ-CONTRACTION ON COMPLETE b-METRIC SPACES

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Abstract. In this paper, we introduce a new notion of generalized θ-ϕ-contraction and establish some results of fixed point for such mappings in complete b-metric space.

1. Introduction

The Banach contraction principle is a fundamental result in fixed point theory [2]. Due to its importance, various mathematics studied many interesting extensions and generalizations, (see [7]). In 2014, Jleli and Samet [5] analyzed a generalization of the Banach fixed point theorem on generalized metric spaces in a new type of contraction mappings called θ -contraction (or JS -contraction) and proved a fixed point result in generalized metric spaces. This direction has been studied and generalized in different spaces and various fixed point theorems have been developed (see [6]). Many generalizations of the concept of metric spaces are defined and some fixed point theorems were proved in these spaces. In particular, b -metric spaces were introduced by Bakhtin [1] and Czerwik [3], in such a way that triangle inequality is replaced by the b-triangle inequality: $d(x,y) \leq s(d(x,z)+d(z,y))$ for all pairwise distinct points x, y, z and $s \geq 1$. Any metric space is a b-metric space but in general, bmetric space might not be a metric space.

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Various fixed point results were established on such spaces. For more information on b -metric spaces and b-metric-like spaces, the readers can refer to (see [4]).

Very recently, Zheng et al. [8] introduced a new concept of θ-ϕ-contraction and established some fixed point results for such mappings in complete metric space and generalized the results of Brower and Kannan.

In this paper, we introduce a new notion of generalized θ-ϕ-contraction and establish some results of fixed point for such mappings in complete b-metric space. The results presented in the paper extend the corresponding results of Kannan and Reich on b-rectangular metric space.

Definition 1.1. Let X be a nonempty set, $s \geq 1$ be a given real number, and let $d : X \times X \to [0, +\infty[$ be a function such that for all $x, y \in X$ and all distinct points $u, v \in X$, each distinct from x and y:

1. $d(x, y) = 0$, if only if $x = y$;
2. $d(x, y) = d(y, x)$;
3. $d(x, y) \leq s[d(x, z) + d(z, y)]$, ($b$-rectangular inequality).

Then (X, d) is called a b-metric space.

Definition 1.2. Let θ be the family of all functions $\theta :]0, +\infty[\to]1, +\infty[$such that

(θ1) θ is increasing,
(θ2) for each sequence $(x_n) \subset]0, +\infty[$;

$$lim_{n\to 0} x_n = 0 \ if \ and \ only \ if \ lim_{n\to\infty} \theta(x_n) = 1;$$

(θ3) θ is continuous.

In [8], Zheng et al. presented the concept of θ-ϕ-contraction on metric spaces and proved the following nice result.

Definition 1.3. Let ϕ be the family of all functions $\phi : [1, +\infty[\to [1, +\infty[$, such that

(φ1) ϕ is nondecreasing;
(φ2) for each $t \in]1, +\infty[, lim_{n\to\infty} \phi^n(t) = 1$;
(φ3) ϕ is continuous.

Definition 1.4. Let (X, d) be a metric space and $T : X \to X$ be a mapping. Then T is said to be a θ-ϕ-contraction if there exist $\theta \in \Theta$ and $\phi \in \Phi$ such that for any $x, y \in X$,

$$d(Tx, Ty) > 0 \Rightarrow [d(Tx, Ty)] \leq \phi(\theta[N(x, y)]),$$

where

$$N(x, y) = max\{(x, y), d(x, Tx), d(y, Ty)\}.$$

Theorem 1.5. *Let (X, d) be a complete metric space and let $T : X \to X$ be a θ-ϕ-contraction. Then T has a unique fixed point.*

2. MAIN RESULTS

In this paper, using the idea introduced by Zheng et al., we present the concept θ-ϕ-contraction in b-metric spaces and we prove some fixed point results for such spaces.

Definition 2.1. Let (X, d) be a b-metric space with parameter $s > 1$ space and $T : X \to X$ be a mapping.

(1) T is said to be a θ-contraction if there exist $\theta \in \Theta$ and $r \in]0, 1[$ such that

$$d(Tx, Ty) > 0 \Rightarrow \theta[s^3 d(Tx, Ty)] \leq \theta[M(x, y)]^r,$$

where

$$M(x, y) = max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(Tx, y)}{2s^2}\}.$$

(2) T is said to be a θ-ϕ-contraction if there exist $\theta \in \Theta$ and $\phi \in \Phi$ such that

$$d(Tx, Ty) > 0 \Rightarrow \theta[s^3 d(Tx, Ty) \leq \phi[\theta(M(x, y))],$$

where

$$M(x, y) = max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(Tx, y)}{2s^2}\}.$$

(3) T is said to be a θ-ϕ- Kannan-type contraction if there exist $\theta \in \Theta$ and $\phi \in \Phi$ such that for all $x, y \in X$ with $d(Tx, Ty) > 0$, we have

$$\theta[s^3 d(Tx, Ty)) \leq \phi[\theta(\frac{d(x, Tx) + d(y, Ty)}{2})].$$

(4) T is said to be a θ-ϕ-Reich-type contraction if there exist exist $\theta \in \Theta$ and $\phi \in \Phi$ such that for all $x, y \in X$ with $d(Tx, Ty) > 0$, we have

$$\theta[s^3 d(Tx, Ty)) \leq \phi[\theta(\frac{d(x, y) + d(x, Tx) + d(y, Ty)}{3})].$$

Theorem 2.2. *Let (X, d) be a complete b-metric space and $T : X \to X$ be a θ-contraction, i.e, there exist $\theta \in \Theta$ and $r \in]0, 1[$ such that for any $x, y \in X$, we have*

$$d(Tx, Ty) > 0 \Rightarrow \theta[s^3 d(Tx, Ty)] \leq \theta[M(x, y)]^r.$$

Then T has a unique fixed point.

Corollary 2.3. *Let (X, d) be a complete b-metric space and $T : X \to X$ be a given mapping. Suppose that there exist $\theta \in \Theta$and $k \in]0, 1[$ such that for any $x, y \in X$, we have*

$$d(Tx, Ty) > 0 \Rightarrow \theta[s^3 d(Tx, Ty)] \leq [\theta(d(x, y))]^k.$$

Theorem 2.4. *Let (X, d) be a complete b-metric space and $T : X \to X$ be a mapping. Suppose that there exist $\theta \in \Theta$ and $\phi \in \Phi$ such that for all $x, y \in X$,*

$$d(Tx,Ty) > 0 \Rightarrow \theta[s^3 d(Tx,Ty)] \leq \phi[\theta(M(x,y))]$$

where

$$M(x,y) = max\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2s^2}d(y,Tx)\}.$$

Then T has a unique fixed point.

It follows from Theorem 2.4 that we obtain the followed fixed point theorems for θ-ϕ-Kannan-type contraction and θ-ϕ-Reich-type contraction. The results presented in the paper improve and extend the corresponding results due to Kannan-type contraction and Reich-type contraction on rectangular b-metric space.

Theorem 2.5. *Let (X,d) be a complete b-metric space and $T : X \to X$ be a Kannan-type contraction. Then T has a unique fixed point.*

Theorem 2.6. *Let (X,d) be a complete b-metric space and $T : X \to X$ be a Reich-type contraction. Then T has a unique fixed point.*

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