HAHN-BANACH THEOREM BEFORE BANACH

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ABSTRACT. In this talk we review the early development of Hahn-Banach theorem before Banach gave the final form of this theorem.

1. Introduction

One of the most important results in Functional analysis is the Hahn-Banach theorem and its consequences. Before Banach, some of mathematicians had obtained the result for some of special classical spaces such as L^p and $C[a, b]$. In this talk we review their efforts. In translation and rewriting their theorems, I tried to be close to their notations.

2. Schmidt and the first steps

The idea of generalizing finite dimensional euclidean spaces to spaces with infinite dimension was noticed at the beginning of the 20th century. Inspired by Hilbert's works, Schmidt in 1908 made the first systematic study of sequence space ℓ^2 [5]. In that paper Schmidt considered numerical sequences whose sum of squares of the absolute value of their terms is finite. Next he defined inner product (without conjugate on the second component and so to obtain the inner product he used $(A, \overline{B})$), norm and orthogonality as usual and deduced Bessel's equation. Next he introduced convergence in norm (which he named strong convergence: *Starken convergenz*) and

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GramSchmidt process. After these preliminary steps in chapter 2 he considered the system of linear equations with infinite unknowns. After studying homogenous equations, he considered nonhomogenous system of equations $(\overline{A_n}, Z) = c_n$ where for each n, $A_n \in \ell^2$ and $c_n \in \mathbb{C}$ and A_n's are independent. By Gram-Schmidt process, he transformed A_n to orthonormal B_n and the system of equations is transformed to $(\overline{B_n}, Z) = g_n$ and by application of Bessel's equation he proved the following:

Theorem 2.1. *The necessary and sufficient condition for the solvability of the equation is the convergence of the series* $\sum |g_n|^2$, *in other words* $g_n \in \ell^2$.

In fact, The above problem is related to extension of a functional, because since ${\ell^2}^* = \ell^2$, hence solvability of the above equation is equivalent to existence of $Z \in \ell^2$, such that $Z(\overline{A_n}) = c_n$. But Schmidt did not go further and the next step was taken by Riesz.

3. Riesz and Helly

In 1909, Riesz introduced the spaces $L^p[a,b]$[3] and obtained most of the classical results about these spaces. But the most relevant result to Hahn-Banach theorem that was proved in this paper was:

Theorem 3.1. *A finite or countably infinite system of linear integral equations*

$$\int_a^b f_i(x)\xi(x)dx = c_i, (i = 1, 2, ...)$$

whose coefficient functions $f_i(x)$ *belong to* $L^{\frac{p}{p-1}}$, *has a solution* ξ *with condition*

$$\int_a^b |\xi(x)|^p dx \le M^p$$

if and only if for each n *and each complex numbers* μ_i,

$$|\sum_{i=1}^{n} \mu_i c_i|^{\frac{p}{p-1}} \le M^{\frac{1}{p-1}} \int_a^b |\sum_{i=1}^{n} \mu_i f_i(x)|^{\frac{p}{p-1}} dx.$$

This theorem is the first form of Hahn-Banach theorem in special case $X = L^p[a,b]$ and $X^* = L^q[a,b]$ where $q = \frac{p}{p-1}$ is the conjugate of p.

In 1911, Riesz proved a similar result for $C[a,b]$ [4].

Theorem 3.2. *The system of linear integral equations*

$$\int_a^b f_k(x)d\alpha(x) = c_k, (k = 1, 2, ...)$$

has a solution α *if and only if there exists a number* M *such that for any* μ_k

$$|\sum_{k=1}^{n} \mu_k c_k| \le M \times Max|\sum_{k=1}^{n} \mu_k f_k(x)|.$$

Total variation of the solution α *is less than or equal with* M.

In 1912 Helly reproved some results of Riesz about $C[a,b]$ [2], specially the above theorem but with a different approach. To obtain the above theorem, Helly first proved a lemma that is similar to modern proof of the Hahn-Banach theorem:

Theorem 3.3. *If for any μ_i's, the inequality*

$$|\sum_{i=1}^{n} \mu_i \gamma_i| \leq M|\sum_{i=1}^{n} \mu_i g_i(x)|$$

holds where for $i = 1, ..n$, γ_i's are fixed real numbers and $g_i \in C[a,b]$, then for each $g_{n+1} \in C[a,b]$, there exists γ_{n+1} such that for any choice of μ_i's, we have

$$|\sum_{i=1}^{n+1} \mu_i \gamma_i| \leq M|\sum_{i=1}^{n+1} \mu_i g_i(x)|.$$

4. Hahn

At last in 1927, Hans Hahn gave the final form of the theorem for real complete normed spaces [1]. But he used norm instead of sublinear functional. In his notation, a linear functional f on a normed space $\Re$ with norm D has slope M, if for each $x \in \Re$,

$$|f(x)| \leq MD(x).$$

Theorem 4.1. *Let $\Re_0$ be a complete linear subspace of $\Re$ and $f_0(x)$ a linear functional on $\Re_0$ of slope M. Then there is a linear functional $f(x)$ on $\Re$ of slope M which coincides with $f_0(x)$ on $\Re_0$.*

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