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MULTIPLICITY OF WEAK SOLUTIONS FOR AN ANISOTROPIC ELLIPTIC SYSTEM

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Abstract. Here, by using variational methods, the multiplicity of weak solutions for a system of problems including the anisotropic $\overrightarrow{p}(x)$-Laplacian operator is proved.

1. Introduction

Anisotropic $\overrightarrow{p}-$Laplacian operator

$$\Delta_{\overrightarrow{p}(x)}u = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(|\frac{\partial u}{\partial x_i}|^{p_i(x)-2}\frac{\partial u}{\partial x_i}\right),$$

$\overrightarrow{p} = (p_1, \cdots, p_N)$, with a complex structure that behaves differently in different directions of space, has been the focus of many authors in recent years [3, 4]. This operator is used in equations that descriptions electromagnetic fields, the plasma physics and elastic mechanics.

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In this paper, using variational methods, we examine the existence and multiplicity of weak solutions for anisotropic system

$$\begin{cases} -\Delta_{\overrightarrow{p}(x)}u+\sum_{i=1}^{N} a_1(x)|u|^{p_i(x)-2}u=\lambda F_u(x,u,v)+\mu G_u(x,u,v) & \text{in } \Omega,\\ -\Delta_{\overrightarrow{p}(x)}v+\sum_{i=1}^{N} a_2(x)|v|^{p_i(x)-2}v=\lambda F_v(x,u,v)+\mu G_v(x,u,v) & \text{in } \Omega,\\ \frac{\partial u}{\partial \nu}=\frac{\partial v}{\partial \nu}=0 & \text{On } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N, N \geq 2$, is a non-empty bounded open set with a boundary $\partial\Omega$ of class C^1, ν is the outer unit normal to $\partial\Omega$. $\overrightarrow{p}=(p_1,\cdots,p_N)$ where for $i=1,\cdots,N$, p_is are continuous functions on Ω with $p_i(x)\geq 2$ for all $x\in\Omega$. Also λ,μ are positive parameters, F_ξ, G_ξ denote the partial derivative of F,G with respect to ξ and $F(x,.,.), G(x,.,.)$ are continuously differentiable in $\mathbb{R}^2$ for a.e. $x\in\Omega$. Moreover, for $i=1,2$, functions $a_i(x)$ are true in the following condition:

(A_0)
$$a_i \in L^\infty(\Omega), \quad a_i^0 := ess \inf_{x\in\Omega} a_i(x) > 0.$$

If $T:\Omega\times\mathbb{R}^2\to\mathbb{R}$ then, we suppose following assumption on T:

(T_0) $T:\Omega\times\mathbb{R}^2\to\mathbb{R}$ is measurable in Ω for all $(s,t)\in\mathbb{R}^2$ and $T(x,.,.)$ is C^1 with respect to $(s,t)\in\mathbb{R}^2$ for a.e. $x\in\Omega$ and for each $\theta>0$,

$$\sup_{|(s,t)|\leq\theta}|T_u(.,s,t)|, \sup_{|(s,t)|\leq\theta}|T_v(.,s,t)| \in L^1(\Omega).$$

2. Preliminaries and notations

We start by introducing the anisotropic variable exponent Sobolev spaces. We consider the vectorial function $\overrightarrow{p}:\overline{\Omega}\to\mathbb{R}^N$ with $\overrightarrow{p}(x)=(p_1(x),\cdots,p_N(x))$ that $p_i\in C_+(\overline{\Omega})$ for all $i\in\{1,\cdots,N\}$. We set

$$p^- := \inf_{x\in\Omega} p(x), \qquad p^+ := \sup_{x\in\Omega} p(x),$$

$$\underline{p}=\min\left\{p_i^- : \quad i=1,\cdots,N\right\}, \quad \overline{p}=\max\left\{p_i^+ : \quad i=1,\cdots,N\right\}.$$

The anisotropic variable exponent Sobolev space is defined as follows

$$W^{1,\overrightarrow{p}(x)}(\Omega)=\left\{u\in L^{p_i(x)}(\Omega): \frac{\partial u}{\partial x_i}\in L^{p_i(x)}(\Omega) \text{ for } i=1,\cdots,N\right\},$$

with the norm $\|u\|_{\overrightarrow{p}} := \|u\|_{W^{1,\overrightarrow{p}(x)}(\Omega)} = \sum_{i=1}^N\left(\|\frac{\partial u}{\partial x_i}\|_{p_i}+\|u\|_{p_i}\right)$. The space $(W^{1,\overrightarrow{p}(x)}(\Omega),\|\cdot\|_{\overrightarrow{p}})$ is a separable and reflexive Banach space. We consider the product space

$$X := W^{1,\overrightarrow{p}(x)}(\Omega)\times W^{1,\overrightarrow{p}(x)}(\Omega)$$

which is equipped with the norm $\|(u,v)\| := \|u\|_{\overrightarrow{p}} + \|v\|_{\overrightarrow{p}}$. Define the functionals $\Phi, \Psi_{\lambda,\mu} : X \to \mathbb{R}$, by

$$\Phi(u,v) := \sum_{i=1}^{N}\left(\int_\Omega \frac{1}{p_i(x)}|\frac{\partial u}{\partial x_i}|^{p_i(x)}dx + \int_\Omega \frac{a_1(x)}{p_i(x)}|u|^{p_i(x)}dx\right)$$
$$+ \sum_{i=1}^{N}\left(\int_\Omega \frac{1}{p_i(x)}|\frac{\partial v}{\partial x_i}|^{p_i(x)}dx + \int_\Omega \frac{a_2(x)}{p_i(x)}|v|^{p_i(x)}dx\right), \tag{2.1}$$

and

$$\Psi_{\lambda,\mu}(u,v) := \int_\Omega F(x,u,v)dx + \frac{\mu}{\lambda}\int_\Omega G(x,u,v)dx, \tag{2.2}$$

for any $(u,v) \in X$. set $I_{\lambda,\mu} = \Phi(u,v) - \lambda\Psi_{\lambda,\mu}(u,v)$. To prove the main theorem, we need the following lemma which we have proved in this article.

Lemma 2.1. *set* $U(x) = \sum_{i=1}^{N}\left(\int_\Omega |\frac{\partial u}{\partial x_i}|^{p_i(x)}dx + \int_\Omega a(x)|u|^{p_i(x)}dx\right)$ *for all* $u \in W^{1,\overrightarrow{p}(x)}(\Omega)$. *So, there exist constants* $\beta_1, \beta_2 > 0$ *that*

(i) $\|u\|_{\overrightarrow{p}} \geq 1 \Longrightarrow \beta_1\|u\|^{\underline{p}}_{\overrightarrow{p}} \leq U(x) \leq \beta_2\|u\|^{\overline{p}}_{\overrightarrow{p}}$,

(ii) $\|u\|_{\overrightarrow{p}} \leq 1 \Longrightarrow \beta_1\|u\|^{\overline{p}}_{\overrightarrow{p}} \leq U(x) \leq \beta_2\|u\|^{\underline{p}}_{\overrightarrow{p}}$.

3. MAIN RESULT

In the following, we will state the main theorem.

Theorem 3.1. *Suppose that*

(A_1) *for each* $(x,s,t) \in \Omega \times \mathbb{R}^+ \times \mathbb{R}^+$, $F(x,s,t) \geq 0$;

(A_2) *there exist* $\alpha \in L^\infty(\Omega), \alpha(x) > 0$ *a.e. in* Ω *and* $\gamma_1, \gamma_2 \in C_+$ *with* $0 < \gamma_1(x) \leq \gamma_1^+ < \gamma_2^+ < \frac{\underline{p}}{2}$ *such that*

$$|F(x,s,t)|, |G(x,s,t)| \leq \alpha(x)\left(1 + |s|^{\gamma_1(x)} + |t|^{\gamma_2(x)}\right)$$

for a.e. $x \in \Omega$ *and each* $(s,t) \in \mathbb{R}^2$;

(A_3) *there exist two positive constants* δ *and* τ *such that*

$$c_0^{\underline{p}} C_{\underline{p}}^2 N^{\underline{p}}(a_1^0 + a_2^0) meas(\Omega) \min\{\delta^{\underline{p}}, \delta^{\overline{p}}\} > \min\{1, a_1^0, a_2^0\}\tau^{\underline{p}};$$

(A_4)

$$\frac{\int_\Omega \sup_{|(s,t)|\leq\tau} F(x,s,t)dx}{\tau^{\underline{p}}} < \frac{\underline{p}\min\{1, a_1^0, a_2^0\}\int_\Omega F(x,\delta,\delta)dx}{\overline{p}C_0^{\underline{p}}C_{\underline{p}}^2 N^{\underline{p}}(\|a_1\|_\infty + \|a_2\|_\infty) meas(\Omega)\max\{\delta^{\overline{p}}, \delta^{\underline{p}}\}};$$

so, for each $\lambda \in \Lambda_{\delta,\tau}$, *given by*

$$\left]\frac{(\|a_1\|_\infty + \|a_2\|_\infty) N meas(\Omega)\max\{\delta^{\overline{p}}, \delta^{\underline{p}}\}}{\underline{p}\int_\Omega F(x,\delta,\delta)dx}, \frac{\min\{1, a_1^0, a_2^0\}\tau^{\underline{p}}}{\overline{p}C_0^{\underline{p}}C_{\underline{p}}^2 N^{\underline{p}-1}\int_\Omega \sup_{|(s,t)|\leq\tau} F(x,s,t)dx}\right[, \tag{3.1}$$

and for every $G : \Omega \times \mathbb{R}^2 \to \mathbb{R}$, *there is* $\varepsilon > 0$ *given by* $\varepsilon = \min\{\mathcal{A}_\tau, \mathcal{B}_\delta\}$, *where*

$$\mathcal{A}_\tau = \frac{\min\{1, a_1^0, a_2^0\}\tau^{\underline{p}} - \lambda\overline{p}C_0^{\underline{p}}C_{\underline{p}}^2 N^{\underline{p}-1}\int_\Omega \sup_{|(s,t)|\leq\tau} F(x,s,t)dx}{\overline{p}C_0^{\underline{p}}C_{\underline{p}}^2 N^{\underline{p}-1}\int_\Omega sup_{|(s,t)|\leq\tau} G(x,s,t)dx},$$

$$\mathcal{B}_\delta = \frac{\lambda\underline{p}\int_\Omega F(x,\delta,\delta)dx - N(\|a_1\|_\infty + \|a_2\|_\infty)meas(\Omega)\max\left\{\delta^{\overline{p}}, \delta^{\underline{p}}\right\}}{\underline{p}\int_\Omega G(x,\delta,\delta)dx},$$

such that for each $\mu \in [0, \varepsilon[$, *the problem* (1.1) *admits at least three distinct weak solutions.*

Proof. Using the critical points theorem of Bonanno and Marano[1], we prove the existence at least three distinct weak solutions for system (1.1). we showed that Φ is coercive and functions Φ and Ψ hold in the conditions of the three critical points theorem of Bonanno, that's mean,

- $\Phi, \Psi_{\lambda,\mu} \in C^1(X, \mathbb{R})$ [2, Lemma 3.4].
- The functional Φ is sequentially weakly lower semicontinuous.
- $\Psi'_{\lambda,\mu} : X \to X^*$ is a compact operator.
- Φ' admits a continuous inverse on X^*.

In the following, for $\delta > 0$, we pick $w(x) := (\delta, \delta)$ for any $x \in \Omega$ and

$$r := \frac{\min\left\{1, a_1^0, a_2^0\right\}}{\overline{p}C_{\underline{p}}^2 N^{\underline{p}-1}} \left(\frac{\tau}{C_0}\right)^{\underline{p}}.$$

We show that for $\lambda \in \left]\frac{\Phi(w)}{\Psi(w)}, \frac{r}{\sup_{(u,v)\in\Phi^{-1}(]-\infty,r[)}\Psi(u,v)}\right[$,

the functional $I_{\lambda,\mu}$ is coercive. Therefore, all the conditions of Bonanno's theorem are satisfied and we can conclude that the functional $I_{\lambda,\mu}$ admits at least three critical points in X which are the weak solutions of system (1.1). □

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