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SOLUTIONS TO A (P,Q)-BIHARMONIC EQUATION WITH NAVIER BOUNDARY CONDITIONS

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Abstract. We study the existence of weak solutions to a (p,q)-biharmonic elliptic equation involving a singular term under Navier boundary conditions, by using variational methods.

1. Introduction

Stationary problems involving singular nonlinearities, as well as the associated evolution equations, describe naturally several physical phenomena and applied economical models. This kind of problems intensively studied in the last decades, specially with the Steklov boundary conditions [4]. In the present paper, we consider the following (p,q)-biharmonic problem

$$\begin{cases} \Delta_p^2 u + \Delta_q^2 u + \theta(x)\frac{|u|^{s-2}u}{|x|^{2s}} = \lambda f(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N (N > 2)$ is a bounded domain with boundary of class C^1 and p,q are positive parameters satisfying the following inequalities $max\{2, N/2\} < q < p < +\infty$. And, $\Delta_r^2 u := \Delta(|\Delta u|^{r-2}\Delta u)$ denotes r-biharmonic operator for $r \in \{p,q\}$; $\theta \in L^\infty(\Omega)$ is a real function with $\inf_{x\in\overline{\Omega}} \theta(x) > 0$; s is a constant such that $1 < s < N/2$; $\lambda > 0$ is a real

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parameter and $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function which holds the following growth condition:

$$|f(x,s)| \leqslant a_1 + a_2|s|^{\gamma-1} \tag{1.2}$$

for $(x,s)\in\Omega\times\mathbb{R}$, where a_1, a_2 and γ are positive constants such that $\gamma\leq p$ a.e. in Ω.

2. Basic definitions and preliminary results

Proposition 2.1. [3] *Let $q\leq p$, a.e. on Ω, then $L^p(\Omega)\hookrightarrow L^q(\Omega)$; moreover, there is a constant k_q such that $|u|_q\leqslant k_q|u|_p$.*

We denote the Sobolev space $W^{k,p}(\Omega)$ for $k=1,2$, by

$$W^{k,p}(\Omega) := \{u\in L^p(\Omega) : D^\alpha u\in L^p(\Omega), |\alpha|\leq k\},$$

that in which $D^\alpha u = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1}x_1\ldots\partial^{\alpha_N}x_N}$ where $\alpha=(\alpha_1,\alpha_2,...,\alpha_N)$ is a multi-index with $|\alpha|=\Sigma_{i=1}^N\alpha_i$. The space $W^{k,p}(\Omega)$ with the norm

$$\|u\|_{k,p} = \Sigma_{|\alpha|\leq k}|D^\alpha u|_p$$

is a Banach separable and reflexive space. We assume that $W_0^{1,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$ which has the norm $\|u\|_{1,p}=|Du|_p$. In what follows, we set

$$X := W_0^{1,p}(\Omega)\cap W^{2,p}(\Omega),$$

endowed with the norm $\|u\| := \int_\Omega |\Delta u|^p dx$.

Remark 2.2. The embedding $X\hookrightarrow C^0(\overline{\Omega})$ is compact; moreover, there exist constant $L>0$ such that $|u|_\infty\leq L\|u\|$, where $|u|_\infty=\sup_{x\in\Omega}u(x)$.

The next is the classical Hardy-Rellich inequality mentioned in [2].

Lemma 2.3. *Let $1<s<\frac{N}{2}$. Then for $u\in W_0^{1,s}(\Omega)\cap W^{2,s}(\Omega)$, one has*

$$\int_\Omega \frac{|u(x)|^s}{|x|^{2s}}dx \leq \frac{1}{\mathcal{H}}\int_\Omega |\Delta u(x)|^s dx,$$

where $\mathcal{H} := (\frac{N(s-1)(N-2p)}{s^2})^s$.

Definition 2.4. We say that function $u\in X$ is a weak solution of Problem (1.1) if $u=\Delta u=0$ on $\partial\Omega$ and

$$\int_\Omega |\Delta u|^{p-2}\Delta u\Delta v dx + \int_\Omega |\Delta u|^{q-2}\Delta u\Delta v dx + \int_\Omega \theta(x)\frac{|u|^{s-2}}{|x|^{2s}}uv dx - \lambda\int_\Omega f(x,u)v dx = 0$$

for every $v\in X$.

In the sequel, we put

$$\delta(x) = sup\{\delta>0 : B(x,\delta)\subseteq\Omega\} \quad \text{and} \quad R := sup_{x\in\Omega}\delta(x).$$

Obviously, there exists $x^0=(x_1^0,\cdots,x_N^0)\in\Omega$ such that $B(x^0,R)\subseteq\Omega$.

3. Existence result

Let $\Phi : X \to \mathbb{R}$ be a functional defined by

$$\Phi(u) = \frac{1}{p}\int_\Omega |\Delta u|^p dx + \frac{1}{q}\int_\Omega |\Delta u|^q dx + \frac{1}{s}\int_\Omega \theta(x)\frac{|u(x)|^s}{|x|^{2s}}dx,$$

Remark 3.1. Under the above assumptions, we gain

$$\frac{1}{p}\|u\|^p \le \Phi(u) \le K(\|u\|^p + \|u\|^s)$$

where $K = \max\{\frac{2}{s}, \frac{2|\theta|_\infty}{\mathcal{H}s}\}$.

Φ is continuously Gâteaux differentiable functional; moreover,

$$\langle \Phi'(u), v\rangle = \int_\Omega (|\Delta u|^{p(x)-2}\Delta u\Delta v + |\Delta u|^{q(x)-2}\Delta u\Delta v + \theta(x)\frac{|u(x)|^{s-2}uv}{|x|^{2s}})dx$$

for $u, v \in X$ (see [5]). Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function with the growth condition (1.2) and define $F(x,t) := \int_0^t f(x,s)ds$. Then the functional $\Psi : X \to \mathbb{R}$ with $\Psi(u) := \int_\Omega F(x,u(x))dx$ for every $u \in X$ is continuously Gâteaux differentiable with the following compact derivative $\langle \Psi'(u), v\rangle := \int_\Omega f(x,u(x))v(x)dx$, for every u, v in X (see [5]). Now, define $I_\lambda = \Phi - \lambda\Psi$.

Theorem 3.2. *Let $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ be Carathéodory function satisfy* (1.2)*. Assume that there exist $r > 0$ and $\delta > 0$ such that*

$$K\Big((\frac{2\delta N}{R^2 - (\frac{R}{2})^2})^p + (\frac{2\delta N}{R^2 - (\frac{R}{2})^2})^s\Big)m\big(R^N - (\frac{R}{2})^N\big) < r,$$

where $m := \frac{\pi^{\frac{N}{2}}}{\frac{N}{2}\Gamma(\frac{N}{2})}$ is the measure of unit ball of $\mathbb{R}^N$ and Γ is the Gamma function. Then for each $\lambda \in]A, B[$, where

$$A := \frac{K\Big((\frac{2\delta N}{R^2-(\frac{R}{2})^2})^p + (\frac{2\delta N}{R^2-(\frac{R}{2})^2})^s\Big)m\big(R^N - (\frac{R}{2})^N\big)}{|\Omega|\big(a_1 L(pr)^{\frac{1}{p}} + \frac{a_2}{\gamma}L_\gamma^\gamma(pr)^{\frac{\gamma}{p}}\big)},$$

and

$$B := \frac{r}{|\Omega|\big(a_1 L(pr)^{\frac{1}{p}} + \frac{a_2}{\gamma}L^\gamma(pr)^{\frac{\gamma}{p}}\big)},$$

Problem (1.1) *admits at least one non-trivial weak solution.*

Proof. For the given $\lambda > 0$, the functional I_λ satisfies the $(P.S.)^{[r]}$ condition. Let the function $w_\lambda \in X$ be defined by

$$w_\lambda(x) := \begin{cases} 0 & x \in \Omega \setminus B(x^0, R), \\ \delta & x \in B(x^0, \frac{R}{2}), \\ \frac{\delta}{R^2-(\frac{R}{2})^2}(R^2 - \sum_{i=1}^N (x_i - x_i^0)^2) & x \in B(x^0, R) \setminus B(x^0, \frac{R}{2}), \end{cases} \tag{3.1}$$

where $x = (x_1, \cdots, x_N) \in \Omega$. Then,

$$\sum_{i=1}^{N} \frac{\partial^2 w}{\partial x_i^2}(x) = \begin{cases} 0 & x \in \big(\Omega \setminus B(x^0, R)\big) \cup B(x^0, \frac{R}{2}), \\ -\dfrac{2\delta N}{R^2 - (\frac{R}{2})^2} & x \in B(x_0, R) \setminus B(x^0, \frac{R}{2}). \end{cases}$$

So, by applying Remark 3.1, one has

$$\begin{aligned} &\frac{1}{p^+}(\frac{2\delta N}{R^2 - (\frac{R}{2})^2})^p m(R^N - (\frac{R}{2})^N) \\ &\qquad < \Phi(w) \\ &\qquad \leq K\Big((\frac{2\delta N}{R^2 - (\frac{R}{2})^2})^p + (\frac{2\delta N}{R^2 - (\frac{R}{2})^2})^s\Big) m\big(R^N - (\frac{R}{2})^N\big), \end{aligned}$$

then, we gain $\Phi(w) < r$. Using Remark 3.1, for each $u \in \Phi^{-1}\left((-\infty, 1[\right)$, we have

$$\|u\| \leq [p^+\Phi(u)]^{\frac{1}{p}} \leq (p^+ r)^{\frac{1}{p}}. \tag{3.2}$$

Hence, from (3.2) and (1.2), we deduce

$$\sup_{\Phi(u)<r} \Psi(u) \leq |\Omega|\big(a_1 L(pr)^{\frac{1}{p}} + \frac{a_2}{\gamma} L^\gamma (pr)^{\frac{\gamma}{p}}\big).$$

Then, from boundedness Φ, one has

$$\frac{\Psi(w)}{\Phi(w)} > \frac{|\Omega|\big(a_1 L(pr)^{\frac{1}{p}} + \frac{a_2}{\gamma} L^\gamma (pr)^{\frac{\gamma}{p}}\big)}{K\Big((\frac{2\delta N}{R^2 - (\frac{R}{2})^2})^p + (\frac{2\delta N}{R^2 - (\frac{R}{2})^2})^s\Big) m\big(R^N - (\frac{R}{2})^N\big)}$$

So, by critical points results duo to Bonanno (Theorem 3.4 of [1]), for each $\lambda \in]A, B[$ the functional I_λ has at least one non-zero critical point which is the weak solution of Problem (1.1). □

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