A CONDITIONAL OPERATOR ON C^*-ALGEBRAS

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ABSTRACT. In this note, we introduce a lower triangular conditional operator on a unital C^*-algebra $\mathcal{A}$.

1. INTRODUCTION

A linear mapping $E : \mathcal{A} \to \mathcal{B}$ is called a projection if $E(b) = b$ for every $b \in \mathcal{B}$. In this case $E^2 = E$ and $\|E\| \geq 1$. Tomiyama in [8] prove that if E is a projection of norm 1 from $\mathcal{A}$ onto $\mathcal{B}$, then E is positive, $E(a^*)E(a) \leq E(a^*a)$ and $\mathcal{B}$-linear, that is, $E(b_1ab_2) = b_1E(a)b_2$ for all $a \in \mathcal{A}$ and $b_1, b_2 \in \mathcal{B}$. A $\mathcal{B}$-linear projection $E : \mathcal{A} \to \mathcal{B}$ which is also a positive mapping, is called a conditional expectation([1, 2, 4, 5, 6, 7, 8]).

Let $a, b \in \mathcal{A}$ and $\alpha \in \mathbb{C}$. We denote by L_a the left multiplication operator on $\mathcal{A}$. Define the linear operator $T_a : \mathcal{A} \to \mathcal{A}$ by $T_a(x) = E(a)x + aE(x) - E(a)E(x)$, where $E : \mathcal{A} \to \mathcal{B}$ is a conditional expectation operator. Each $a \in \mathcal{A}$ can be written uniquely as $a = a_1 + a_2$ where $a_1 = E(a) \in \mathcal{B}$ and $a_2 = a - E(a) \in \mathcal{N}(E)$, because $\mathcal{A} = \mathcal{B} \oplus \mathcal{N}(E)$. It follows that $T_a = L_{a_1} + L_aE - L_{a_1}E = L_{a_1} + L_{a_2}E$. Thus, $\alpha T_a + T_b = T_{\alpha a+b}$, $T_a(\mathcal{N}(E)) \subseteq \mathcal{N}(E)$ and $\|T_a\| \leq 3\|a\|$. When $e = 1$ then $T_1 = I$, the identity operator. The matrix representation of T_a with respect to the decomposition $\mathcal{A} = \mathcal{B} \oplus \mathcal{N}(E)$ is

$$T_a = \left[\begin{array}{cc} L_{a_1} & 0 \\ L_{a_2} & L_{a_1} \end{array} \right],$$

where $a = a_1 + a_2$. Put $a \star b = a \star_E b = T_a(b)$. Then $a \star b = a_1b + ab_1 - a_1b_1 = a_1b_1 + (a_1b_2 + a_2b_1)$. So $(a \star b)_1 = a_1b_1$ and $(a \star b)_2 = a_1b_2 + a_2b_1$. It follows that

$$T_aT_b = \begin{bmatrix} L_{a_1} & 0 \\ L_{a_2} & L_{a_1} \end{bmatrix} \begin{bmatrix} L_{b_1} & 0 \\ L_{b_2} & L_{b_1} \end{bmatrix} = \begin{bmatrix} L_{(a\star b)_1} & 0 \\ L_{(a\star b)_2} & L_{(a\star b)_1} \end{bmatrix} = T_{a\star b}.$$

Put $\mathcal{K} = \mathcal{K}(E) = \{T_a = L_{a_1} + L_{a_2}E : a \in \mathcal{A}\}$. Then $\mathcal{K}$ is a subalgebra of $B(\mathcal{A})$, the Banach algebra of all bounded and linear maps defined on $\mathcal{A}$ and with values in $\mathcal{A}$. Note that the mapping $\mathcal{T} : \mathcal{A} \to \mathcal{K}$ given by $\mathcal{T}(a) = T_a$ is linear with $\|\mathcal{T}\| \leq 3$ and $\mathcal{T}(a \star b) = \mathcal{T}(a)\mathcal{T}(b)$ for all $a, b \in \mathcal{A}$.

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2. Characterizations

Let E_1, E_2 be two distinct conditional expectations from $\mathcal{A}$ onto $\mathcal{B}$. Then it is easy to check that $G := E_1 + E_2 - I$ is invertible. Since $E_1E_2 = E_2$ and $E_2E_1 = E_1$, then we have $E_1G = E_2 = GE_2$, $E_2G = E_1 = GE_1$ and $(I - E_2)(I - E_1) = I - E_2$.

Proposition 2.1. *For $a \in \mathcal{A}$, let $T_a \in \mathcal{K}(E_1)$ and $S_a \in \mathcal{K}(E_2)$. Then there is an invertible operator G on $\mathcal{A}$ such that $GT_aG = S_{G(a)}$ and the mapping $\Lambda : T_a \to GT_aG$ is an algebra isomorphism of $\mathcal{K}(E_1)$ onto $\mathcal{K}(E_2)$ which is a homeomorphism.*

Proof. Take $G = E_1 + E_2 - I$. Then G is invertible with $G^{-1} = G$. Recall that for each $a, b \in \mathcal{A}$, $T_a(b) = a \star_{E_1} b = (E_1a)b + a(E_1b) - (E_1a)(E_1b)$ and $S_a(b) = a \star_{E_2} b = (E_2a)b + a(E_2b) - (E_2a)(E_2b)$. Then we have

$$\begin{aligned}(T_aG)(b) &= T_a(E_1b + E_2b - b)\\ &= a \star_{E_1} (E_1b) + a \star_{E_1} (E_2b) - a \star_{E_1} b\\ &= a(E_2b) + (E_1a)(E_1b) - (E_1a)b,\end{aligned}$$

and

$$\begin{aligned}(GT_aG)(b) &= (E_1 + E_2 - I)[a(E_2b) + (E_1a)(E_1b) - (E_1a)b]\\ &= E_2b - a(E_2b) + (E_1a)b.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}S_{Ga}(b) &= (Ga) \star_{E_2} b = E_2(Ga)b + (Ga)(E_2b) - E_2(Ga)E_2(b)\\ &= (E_1a)b + (E_2a)(E_2b) - a(E_2b).\end{aligned}$$

Thus, $GT_aG = S_{Ga} \in \mathcal{K}(E_2)$. Also, $\Lambda(T_aT_b) = \Lambda(T_a)\Lambda(T_b)$. So, Λ is a continuous algebra isomorphism and $\Lambda^{-1}(S_b) = GS_bG$ is also continuous with respect to any of the operator topologies. □

Proposition 2.2. *Let $a \in \mathcal{A}$. If a_1 has a left inverse, then T_a is injective. Moreover, if $\mathcal{B}$ has a right invertible element, then the mapping $\mathcal{T} : \mathcal{A} \to \mathcal{K}$ given by $\mathcal{T}(a) = T_a$ is injective.*

Proof. Let $T_a(b) = 0$ for some $b \in \mathcal{A}$. Then $a_1b_1 = -(a_1b_2 + a_2b_1) \in \mathcal{A} \cap \mathcal{N}(E) = \{0\}$ and so $b_1 = 0$. It follows that $a_1b = 0$ and hence $b = 0$. Now, let $b_0 \in \mathcal{B}$ is a right invertible element and let $T_a(b) = a_1b + ab_1 - a_1b_1 = 0$ for all $b \in \mathcal{A}$. Take $b = e$. Then $ae = 0$ and so $a_1 = E(ae) = 0$. Thus, $a_2b_1 = 0$ for all $b_1 \in \mathcal{B}$. Take $b_1 = b_0$. Then $a_2 = 0$. Consequently, $a = 0$. □

Proposition 2.3. *Let $S_0(\mathcal{A}|\mathcal{B}) = \{x \in \mathcal{A} : \mathcal{A}ex \subseteq \mathcal{B}\}$. Then the following assertions hold.*

(i) $\mathcal{N}e + e\mathcal{N} + \mathcal{B} = \vee_{a \in A}\mathcal{R}(T_a)$, *where $\vee$ denotes the algebraic span.*

(ii) $\cup_{a \in A}T_a(S_0) \subseteq \mathcal{B}$.

(iii) $\mathcal{N} \subseteq \cap_{a \in \mathcal{N}}\mathcal{N}(T_a)$. *Moreover, if $\mathcal{N}$ has a left invertible element, then $\mathcal{N} = \cap_{a \in \mathcal{N}}\mathcal{N}(T_a)$.*

Proof. (i) Let $a, x \in A$. Then $T_a(x) = (a_2x_1)e + e(a_1x_2) + a_1x_1 \in \mathcal{N}e + e\mathcal{N} + \mathcal{B}$ and hence $\vee_{a\in A}\mathcal{R}(T_a) \subseteq \mathcal{N}e + e\mathcal{N} + \mathcal{B}$. Conversely, let $k \in \mathcal{N}$ and $b \in \mathcal{B}$. Since $ek = T_1(k)$, $ke = T_k(1)$ and $b = T_b(e)$, then $\mathcal{N}e + e\mathcal{N} + \mathcal{B} \subseteq \vee_{a\in A}\mathcal{R}(T_a)$.

(ii) Let $a \in A$ and $x \in S_0$. Then $\{ex, aex\} \subset \mathcal{B}$, $x_1 = E(x) = E(ex) = ex$ and so $T_a(x) = a_1x + ax_1 - a_1x_1 = a_1ex + aex - a_1ex = aex \in \mathcal{B}$. Thus, $\cup_{a\in\mathcal{A}}T_a(S_0) \subseteq \mathcal{B}$.

(iii) Let $\{a, x\} \subset \mathcal{N}$. Then $a_1 = 0 = x_1$, $T_a(x) = 0$ and so $x \in \mathcal{N}(T_a)$ for all $a \in \mathcal{N}$. Now let $x \in \cap_{a\in\mathcal{N}}\mathcal{N}(T_a)$ and for some $a_2 \in \mathcal{N}$, there is an element $a_0 \in \mathcal{A}$ such that $a_0a_2 = 1$. Then $a_2x_1 = T_{a_2}(x) = 0$ and hence $x_1 = a_0a_2x_1 = 0$. Thus, $x = x_2 \in \mathcal{N}$. □

Proposition 2.4. *Let $a, b \in A$. Then the equation $T_aX = T_b$ has a solution in $\mathcal{K}$ whenever a_1 has a left inverse.*

Proof. Let $a_0a_1 = 1$ and $X = T_x$ for some $a_0, x \in \mathcal{A}$. According to the matrix form of $T_aT_x = T_b$, we have

$$\begin{bmatrix} L_{a_1x_1} & 0 \\ L_{a_2x_1+a_1x_2} & L_{a_1x_1} \end{bmatrix} = \begin{bmatrix} L_{b_1} & 0 \\ L_{b_2} & L_{b_1} \end{bmatrix}.$$

It follows that $a_1x_1 = b_1$ and $a_2x_1 + a_1x_2 = b_2$. Thus, $x_1 = a_0b_1$ and $a_1x_2 = b_2 - a_2x_1$. Then $x_2 = a_0b_2 - a_0a_2a_0b_1$ and hence $x = x_1 + x_2 = a_0b + a_0a_2a_0b_1$. □

It has been shown in [3, Lemma 1.4, Proposition 3.1] that $s\mathcal{N} = \mathcal{N}s = 0$ and $\|b + S_1\|_{\frac{\mathcal{B}}{S_1}} = \|L_b\|_{\mathcal{N}\to\mathcal{N}}$, for all $s \in S_1$ and $b \in \mathcal{B}$. Using these, we have the following result.

Proposition 2.5. *Let $\|I - E\| \leq 1$. Then*

$$\|L_{a_1}\|_{\mathcal{N}\to\mathcal{N}} \leq \inf_{k\in\mathcal{N}} \|T_{a+k}\| \leq \|T_{a_1}\| \leq \|L_{a_1}\|_{\mathcal{B}\to\mathcal{B}} + \|L_{a_1}\|_{\mathcal{N}\to\mathcal{N}}.$$

Proof. Let $s \in S_1$ and $x \in \mathcal{A}$ with $\|x\| = 1$. Since E is a contraction, then we have $\|x_1\| = \|E(x)\| \leq \|E\| \; \|x\| \leq \|x\| = 1$ and $\|x_2\| = \|(I - E)x\| \leq \|I - E\| \; \|x\| \leq 1$. Then we get that

$$\begin{aligned} \|T_{a_1}x\| &= \|a_1x_1 + a_1x_2\| \leq \|a_1x_1\| + \|a_1x_2\| \\ &= \|a_1x_1\| + \|(a_1 + s)x_2\| \leq \sup_{x_1\in B} \|a_1x_1\| + \|a_1 + s\| \\ &= \|L_{a_1}\|_{B\to B} + \inf \|a_1 + s\| = \|L_{a_1}\|_{B\to B} + \|a_1 + S_1\|_{\frac{B}{s_1}}. \end{aligned}$$

Thus, $\|T_{a_1}\| \leq \|L_{a_1}\|_{B\to B} + \|L_{a_1}\|_{\mathcal{N}\to\mathcal{N}}$. Also, we have

$$\inf_{k\in\mathcal{N}} \|T_{a+k}\| \leq \|T_{a+(a_1-a)}\| = \|T_{a_1}\|.$$

On the other hand, $\|T_{a+k}\| = \|T_{a_1+(a_2+k)}\| = \sup_{\|x\|=1} \|a_1x + (a_2 + k)x_1\| \geq \sup_{\|x\|=1} \|a_1x_2\| = \|L_{a_1}\|_{\mathcal{N}\to\mathcal{N}}$. Hence, $\inf_{k\in\mathcal{N}} \|T_{a+k}\| \geq \|L_{a_1}\|_{\mathcal{N}\to\mathcal{N}}$. □

Proposition 2.6. *$\mathcal{K}$ is closed in the norm operator topology.*

Proof. Let $\{T_{a_n}\} \subseteq \mathcal{K}$ and $\|T_{a_n} - T\| \to 0$, for some $T \in B(\mathcal{A})$. Then we have

$$\lim_{n\to\infty} T_{a_n} = \lim_{n\to\infty} \begin{bmatrix} L_{a_{n1}} & 0 \\ L_{a_{n2}} & L_{a_{n1}} \end{bmatrix} = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} = T$$

where $a_{n1} = E(a_n)$ and $a_{n2} = a_n - E(a_n)$. Since $T_{a_n}(\mathcal{N}) \subseteq \mathcal{N}$ then $T(\mathcal{N}) \subseteq \mathcal{N}$, and so $T_2 = 0$. Further, $\lim_{n\to\infty} \|T_{a_{n1}} - T_1\|_{\mathcal{B}\to\mathcal{B}} = 0$ implies that $\lim_{n\to\infty} a_{n1} = \lim_{n\to\infty} a_{n1}e = T_1 e = ETEe = E(Te)$, and so $T_1 x_1 = \lim_{n\to\infty} a_{n1}x_1 = E(Te)x_1$ for all $x_1 \in \mathcal{B}$. Thus, $T_1 = L_{E(Te)}$. Likewise, for each $x_2 \in \mathcal{N}$ we have $T_4 x_2 = \lim_{n\to\infty} a_{n1}x_2 = E(Te)x_2$ and hence $T_4 = L_{E(Te)}$. Moreover, since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} T_{a_n}1 = T1$, then for each $x_1 \in \mathcal{B}$ we obtain that $T_3 x_1 = \lim_{n\to\infty} a_{n2}x_1 = \lim_{n\to\infty}(a_n - a_{n1})x_1 = (T1 - E(Te))x_1$. Cosequently, $T1 - E(Te) \in \mathcal{N}$, $T_3 = L_{T1-E(Te)}$ and

$$T = \begin{bmatrix} L_{E(Te)} & 0 \\ L_{T1-E(Te)} & L_{E(Te)} \end{bmatrix} \in \mathcal{K}.$$

□

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