The 7th Seminar on Functional Analysis and its Applications
1-2 March 2023
Imam Khomeini International University
Qazvin, Iran

COUPLED FIXED POINT THEOREMS IN ORDERED M-METRIC SPACES

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Abstract. Using control functions, we improve and extend some results of coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces by Lakshmikantam and Ćirić [2] to ordered M-metric spaces.

1. Introduction

Partial metric spaces, which are generalizations of metric spaces, introduced by S. G. Mathews [3] as a part of the study of denotational semantics of data flow networks and he gave a Banach fixed point result for these spaces. After that, In 2014 Asadi *et al.* [1] introduced the M-metric space which extends p-metric space, by some of certain fixed point theorems obtained therein, many authors proved fixed point theorems on M-metric spaces, see [4]. Lakshmikantam and Ćirić in [2], gave some interesting fixed point results on ordered metric spaces, but there exist two error in that paper. In this paper, using a class of control functions, we extend main results of [2] to complete M-metric spaces. We mention two errors of [2] as well.

Definition 1.1. ([1]) Let X be a non empty set. A function $m : X \times X \to \mathbb{R}^+$ is called a m-metric if the following conditions are satisfied:

(m1) $m(x,x) = m(y,y) = m(x,y) \iff x = y$,

2020 *Mathematics Subject Classification. Primary 47H10; Secondary 54H25*

Key words and phrases. Fixed point, Partial metric space, M-metric space, Coupled fixed point.

(m2) $m_{xy} \leq m(x,y)$,
(m3) $m(x,y) = m(y,x)$,
(m4) $(m(x,y) - m_{xy}) \leq (m(x,z) - m_{xz}) + (m(z,y) - m_{zy})$.

Where

$$m_{xy} := \min\{m(x,x), m(y,y)\}.$$

Then the pair (X, m) is called an M-metric space.

The following notation is useful in the sequel.

$$M_{xy} := \max\{m(x,x), m(y,y)\}.$$

We note that every p-metric is a m-metric. In the following example we present an example of a m-metric which is not p-metric.

Example 1.2. ([1]) Let $X = \{1, 2, 3\}$. Define

$$m(1,2) = m(2,1) = m(1,1) = 8$$

$$m(1,3) = m(3,1) = m(3,2) = m(2,3) = 7 \quad m(2,2) = 9 \quad m(3,3) = 5$$

so m is m-metric but m is not p-metric. Since $m(2,2) \nleq m(1,2)$ means m is not p-metric. If $D(x,y) = m(x,y) - m_{x,y}$ then $m(1,2) = m_{1,2} = 8$ but it means $D(1,2) = 0$ while $1 \neq 2$ which means D is not metric.

Definition 1.3. Let $(X, \leq)$ be a partially ordered set and $f : X \times X \to X$ and $g : X \to X$. We say f has the mixed g-monotone property if for any $x, y, x_1, x_2, y_1, y_2 \in X$

$$g(x_1) \leq g(x_2) \text{ implies } \quad f(x_1, y) \leq f(x_2, y) \tag{1.1}$$

and

$$g(y_1) \leq g(y_2) \text{ implies } \quad f(x, y_1) \leq f(x, y_2) \tag{1.2}$$

Definition 1.4. An element $(x, y) \in X \times X$ is called a coupled coincidence point of mappings $f : X \times X \to X$ and $g : X \to X$ if

$$f(x,y) = g(x), \quad f(y,x) = g(y).$$

Definition 1.5. Let $f : X \times X \to X$ and $g : X \to X$. We say f and g are commutative if for all $x, y \in X$

$$g(f(x,y)) = f(g(x), g(y)).$$

For simplicity, we denote $g(x)$ by gx.

2. MAIN RESULTS

Definition 2.1. Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be such that $\psi(t) \to 0$ if and only if $t \to 0$, ψ^{-1} is nondecreasing and one to one and $\psi(a+b) \leq \psi(a) + \psi(b)$ (sub-additivity) for $a, b \in \mathbb{R}^+$. We denote the set of these functions by Ψ. For example, $\psi(t) = \alpha t$ and $\psi(t) = e^{\alpha t}$ for $\alpha \geq 0$ belong to Ψ.

Theorem 2.2. *Let $(X, m, \leq)$ be a partially ordered complete M-metric space. Assume that there is a function $\varphi : [0, +\infty) \to [0, +\infty)$ with $\varphi(0) = 0$, $\varphi(t) < t$ and $\lim_{r \to t^+} \varphi(r) < t$ for each $t > 0$ and also suppose $f : X \times X \to X$ and $g : X \to X$ are such that f has the mixed g-monotone property and*

$$\psi(m(f(x, y), f(u, v))) \leq \varphi(\frac{\psi(m(gx, gu)) + \psi(m(gy, gv))}{2}) \tag{2.1}$$

for all $x, y, u, v \in X$ for which $gx \leq gu$ and $gy \geq gv$ and $\psi \in \Psi$. Suppose $f(X \times X) \subseteq g(X)$, g is continuous and commute with f and also suppose either
(a) f is continuous or
(b) X has the following property:

(i) *if for a non-decreasing sequence $m^w(x_n, x) \to 0$, then*

$$gx_n \leq gx \ \text{ for all}\, n. \tag{4}$$

(ii) *if for a non-increasing sequence $m^w(y_n, y) \to 0$, then*

$$gy \leq gy_n \ \text{ for all}\, n. \tag{5}$$

If there exist $x_0, y_0 \in X$ such that

$$g(x_0) \leq f(x_0, y_0) \quad \text{and} \quad g(y_0) \geq f(y_0, x_0),$$

then f and g have a coupled coincidence point.

If $(X, \leq)$ is a partially ordered set, we can endowed $X \times X$ with a partial order as follows:

$$(x_1, y_1) \leq (x_2, y_2) \Longleftrightarrow x_1 \leq x_2 \text{ and } y_2 \leq y_1.$$

The following uniqueness theorem is a generalization of Theorem 2.2 of [5].

Theorem 2.3. *If in Theorem 2.2, we also suppose that for any (x, y) and (x^*, y^*) in $X \times X$, there is $(u, v) \in X \times X$ such that $(f(u, v), f(v, u))$ is comparable to $(f(x, y), f(y, x))$ and $(f(x^*, y^*), f(y^*, x^*))$, then there is a unique point (z, w) such that $z = gz = f(z, w), w = gw = f(w, z)$.*

Remark 2.4. It is noted that, the relation (31) in the proof of Theorem 2.2 of [2], is not correct because in general φ is not non-decreasing.

References

1. M. Asadi, E. Karapınar, and P. Salimi, *New Extension of p-Metric Spaces with Some fixed point Results on M-metric spaces*, Journal of Inequalities and Applications 2014, 2014:18.
2. Lakshmikantham V.,Ćirić L.: Coupled fixed point theorems for nonlinear contractions in partialy ordered metric spaces, Non linear Anal. **70** 4341-4349 (2009).
3. Mathews, S.G.: partial metric topology. Proc. 8th Summer Conference on General topology and Applications Ann New York, Acad Sci: **728**, 183-197 (1994).
4. H. Monfared, M. Azhini and M. Asadi, *Fixed point results on M-metric spaces*, Journal of Mathematical Analysis, **7**(5)(2016), 85-101.
5. Oltra, S., Valero, O: Banach's fixed point theorem for partial spaces. Rend Istit Math Univ Trieste **36**, 17-26 (2004).