The 7th Seminar on Functional Analysis and its Applications
1-2 March 2023
Imam Khomeini International University
Qazvin, Iran

$F(\psi, \varphi)$-CONTRACTIONS ON M-METRIC SPACES

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Abstract. Partial metric spaces were introduced by Matthews in 1994 as a part of the study of denotational semantics of data flow networks. In 2014 Asadi and *et al.* [1] extend the Partial metric spaces to M-metric spaces. In this work, we introduce the class of $F(\psi, \varphi)$-contractions and investigate the existence and uniqueness of fixed points for the new class $\mathcal{C}$ in the setting of M-metric spaces. The theorems that we prove generalize many previously obtained results. We also give some examples showing that our theorems are indeed proper extensions.

1. Introduction

The notion of metric space was introduced by Fréchet [2] in 1906. Later, many authors attempted to generalize the notion of metric space such as pseudo metric space, quasi metric space, semi metric spaces and partial metric spaces. In this paper, we consider another generalization of a metric space, so called M-metric space. This notion was introduced by Asadi *et al.* (see e.g. [1]) to solve some difficulties in domain theory of computer science. Geraghty in 1973 introduced an interesting class of auxiliary function to refine the Banach contraction mapping principle. Let $\mathcal{F}$ denote all functions $\beta : [0, \infty) \to [0, 1)$ which satisfies the condition:

$$\lim_{n\to\infty} \beta(t_n) = 1 \text{ implies } \lim_{n\to\infty} t_n = 0.$$

2020 *Mathematics Subject Classification. Primary 47H10; Secondary 54H25*

Key words and phrases. Fixed point, Partial metric space, M-metric space, Coupled fixed point.

By using the function $\beta \in \mathcal{F}$ Geraghty [3] proved the following remarkable theorem.

Theorem 1.1. (Geraghty [3]) *Let (X,d) be a complete metric space and $T : X \to X$ be an operator. Suppose that there exists $\beta : [0,\infty) \to [0,1)$ satisfying the condition,*

$$\beta(t_n) \to 1 \ \textit{implies} \ t_n \to 0$$

If T satisfies the following inequality

$$d(Tx,Ty) \leq \beta(d(x,y))d(x,y), \ \textit{for any } x,y \in X, \tag{1.1}$$

then T has a unique fixed point.

In 2014 Asadi *et al.* [1] introduced the M-metric space which extends partial metric space [4], by some of certain fixed point theorems obtained therein, and they have given a theorem that its proof is still open as follows.

Theorem 1.2. *Let (X,m) be a complete M-metric space and $T : X \to X$ be mapping satisfying:*

$$\exists k \in [0,\frac{1}{2}) \quad \text{such that} \quad m(Tx,Ty) \leq k(m(x,Ty) + m(y,Tx)) \quad \forall x,y \in X.$$

Admissible mappings have been defined recently by Samet et al [**?**] and employed quite often in order to generalize the results on various contractions. We state next the definitions of α-admissible mapping and triangular α-admissible mappings.

Definition 1.3. Let $\alpha : X \times X \to [0.\infty)$. A self-mapping $T : X \to X$ is called α-admissible if the condition

$$\alpha(x,y) \geq 1 \Rightarrow \alpha(Tx,Ty) \geq 1, \tag{1.2}$$

is satisfied for all $x,y \in X$.

Definition 1.4. A mapping $T : X \to X$ is called triangular α-admissible if it is α-admissible and satisfies

$$\alpha(x,y) \geq 1, \alpha(y,z) \geq 1 \Rightarrow \alpha(x,z) \geq 1. \tag{1.3}$$

where $x,y,z \in X$ and $\alpha : X \times X \to [0.\infty)$ is a given function.

In what follows we recall the notion of (triangular) α-orbital admissible, introduced by Popescu [6], that is inspired from [5].

Definition 1.5. [6] For a fixed mapping $\alpha : X \times X \to [0,\infty)$, we say that a self-mapping $T : X \to X$ is an α-orbital admissible if

$$(O1) \ \ \alpha(u,Tu) \geq 1 \Rightarrow \alpha(Tu,T^2u) \geq 1.$$

Let $\mathcal{A}$ be the collection of all α-orbital admissible $T : X \to X$.

In addition, T is called triangular α-orbital admissible if T is α-orbital admissible and

$$(O2) \ \ \alpha(u,v) \geq 1 \text{ and } \alpha(v,Tv) \geq 1 \Rightarrow \alpha(u,Tv) \geq 1$$

Let $\mathcal{O}$ be the collection of all triangular α-orbital admissible $T : M \to M$.

Definition 1.6. ([1]) Let X be a non empty set. A function $m : X \times X \to \mathbb{R}^+$ is called M-metric if the following conditions are satisfied:

(m1) $m(x,x) = m(y,y) = m(x,y) \iff x = y$,
(m2) $m_{xy} \leq m(x,y)$,
(m3) $m(x,y) = m(y,x)$,
(m4) $(m(x,y) - m_{xy}) \leq (m(x,z) - m_{xz}) + (m(z,y) - m_{zy})$.

Where

$$m_{xy} := \min\{m(x,x), m(y,y)\} = m(x,x) \vee m(y,y),$$

Then the pair (X,m) is called a M-metric space.

Definition 1.7. A function $\psi : [0,\infty) \to [0,\infty)$ is called an altering distance function if the following properties are satisfied:

(i) ψ is non-decreasing and continuous,
(ii) $\psi(t) = 0$ if and only if $t = 0$.

Remark 1.8. We let Ψ denote the class of the altering distance functions.

Definition 1.9. An ultra altering distance function is a continuous, non-decreasing mapping $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi(t) > 0$ for $t > 0$ and $\varphi(0) \geq 0$.

Remark 1.10. We let Φ denote the class of the ultra altering distance functions.

Definition 1.11. A mapping $F : [0,\infty)^2 \to \mathbb{R}$ is called C-*class* function if it is continuous and satisfies following axioms:

(1) $F(s,t) \leq s$;
(2) $F(s,t) = s$ implies that either $s = 0$ or $t = 0$; for all $s,t \in [0,\infty)$.

Note for some F we have that $F(0,0) = 0$.
We denote C-class functions as $\mathcal{C}$.

2. MAIN RESULTS

Definition 2.1. Let (X,m) be an M-metric space, and let $T : X \to X$ be a given mapping. We say that T is $F(\alpha,\psi)$-contractive mapping if there exist $\psi \in \Psi$, $\varphi \in \Phi$ and $F \in \mathcal{C}$ such that

$$\psi(m(Tx,Ty)) \leq F(\psi(m(x,y)), \varphi(m(x,y))), \tag{2.1}$$

Definition 2.2. Let (X,m) be an M-metric space, and let $T : X \to X$ be an α-admissible mapping. We say that T is an α-admissible $F(\alpha,\psi)$-contractive mapping if there exist $\psi \in \Psi$, $\varphi \in \Phi$ and $F \in \mathcal{C}$ such that

$$\alpha(x,y)\psi(m(Tx,Ty)) \leq F(\psi(m(x,y)), \varphi(m(x,y))), \tag{2.2}$$

Definition 2.3. Let (X,m) be an M-metric space, and let $T : X \to X$ be an α-admissible mapping. We say that T is a generalized α-admissible $F(\alpha,\psi)$-contractive mapping if there exist $\psi \in \Psi$, $\varphi \in \Phi$ and $F \in \mathcal{C}$ such that

$$\alpha(x,y)\psi(m(Tx,Ty)) \leq F(\psi(M(x,y)), \varphi(M(x,y))), \tag{2.3}$$

Where $M(x,y) = \max\{m(x,y), m(x,Tx), m(y,Ty)\}$

Theorem 2.4. *(X,m) be a complete M-metric space, and let $T : X \to X$ be a generalized α-admissible $F(\alpha,\psi)$-contractive mapping. and satisfies the following conditions:*
(i) $T \in \mathcal{O}$;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;
(iii) T is continuous.
Then T has a fixed point $v \in X$ and $\{T^n x_0\}$ converges to v.

For the uniqueness of a fixed point of a generalized α-admissible $F(\alpha,\psi)$-contractive mapping, we shall suggest the following hypothesis.
$(*)$ For all $x, y \in \mathrm{Fix}(T)$, we have $\alpha(x,y) \geq 1$.
Here, $\mathrm{Fix}(T)$ denotes the set of fixed points of T.

Theorem 2.5. *Adding condition $(*)$ to the hypotheses of Theorem 2.4, we obtain that v is the unique fixed point of T.*

If we let $\alpha(x,y) = 1$ for all $x, y \in X$, then we get the following Corollaries.

Corollary 2.6. *Let (X,m) be a complete M-metric space, and let $T : X \to X$ be a map. Suppose that the following conditions are satisfied:*

$$\psi(m(Tx,Ty)) \leq F(\psi(M(x,y)), \varphi(M(x,y))), \tag{2.4}$$

Where $M(x,y) = \max\{m(x,y), m(x,Tx), m(y,Ty)\}$, $\psi \in \Psi$, $\varphi \in \Phi$ and $F \in \mathcal{C}$. Then T has unique fixed point $x^* \in X$ and $\{T^n x_0\}$ converges to x^*.

Corollary 2.7. *Let (X,m) be a complete M-metric space and $T : X \to X$ be a self-mapping satisfying*

$$\psi(m(Tx,Ty)) \leq F(\psi(m(x,y)), \varphi(m(x,y))), \tag{2.5}$$

Where $\psi \in \Psi$, $\varphi \in \Phi$ and $F \in \mathcal{C}$. Then T has unique fixed point $v \in X$ and $\{T^n x_0\}$ converges to v.

If we let $F(s,t) = ks$ in Corollary 2.7, we get the following result.

Corollary 2.8. *Let (X,m) be a complete M-metric space, and let $T : X \to X$ be a map. Suppose that the following conditions are satisfied:*

$$\psi(m(Tx,Ty)) \leq k\psi(m(x,y))$$

Then, T has unique fixed point $v \in X$, and $\{T^n x_0\}$ converges to v.

If we let $\psi(t) = t$ in Corollary 2.7, we get the following result.

Corollary 2.9. *Let (X,m) be a complete M-metric space, and let $T : X \to X$ be a map. Suppose that the following conditions are satisfied:*

$$m(Tx,Ty) \leq F(m(x,y), \varphi(m(x,y))),$$

Then, T has unique fixed point $v \in X$, and $\{T^n x_0\}$ converges to v.

If we take $\psi(t) = t$ in Corollary 2.6, we get the following.

Corollary 2.10. *Let (X, m) be a complete M-metric space, and let $T : X \to X$ be a map. Suppose that the following conditions are satisfied:*

$$m(Tx, Ty) \leq F(M(x, y), \varphi(M(x, y))),$$

Where $M(x, y) = \max\{m(x, y), m(x, Tx), m(y, Ty)\}$ and $\varphi \in \Phi, F \in \mathcal{C}$. Then, T has a unique fixed point $v \in X$, and $\{T^n x_0\}$ converges to v.

3. Consequences

By Corollary 2.6 and Corollary 2.7, we obtain the following corollaries as an extension of several known results in the literature.

Corollary 3.1. *Let (X, m) be a complete M-metric space and $T : X \to X$ be a continuous map. Assume that there exists two function $\beta \in \mathcal{F}$ and $\psi \in \Psi$ such that*

$$\psi(m(Tx, Ty)) \leq \beta(\psi(m(x, y)))\psi(m(x, y)), \tag{3.1}$$

for all $x, y \in X$. Then T has unique fixed point.

Corollary 3.2. *Let (X, m) be a complete M-metric space and $T : X \to X$ be a continuous map. Assume that there exists two function $\beta \in \mathcal{F}$ and $\psi \in \Psi$ such that for all $x, y \in X$*

$$\psi(m(Tx, Ty)) \leq \beta(\psi(M(x, y)))\psi(M(x, y)), \tag{3.2}$$

where $M(x, y) = \max\{m(x, y), m(x, Tx), m(y, Ty)\}$. Then T has unique fixed point.

If we let $\psi(t) = t$, we get the following corollary that proved in [?] :

Corollary 3.3. *Let (X, m) be a complete M-metric space and $T : X \to X$ be a continuous map. Assume that there exists a function $\beta \in \mathcal{F}$ such that*

$$m(Tx, Ty) \leq \beta(m(x, y))m(x, y), \tag{3.3}$$

for all $x, y \in X$. Then T has unique fixed point.

Corollary 3.4. *Let (X, m) be a complete M-metric space and $T : X \to X$ be a continuous map. Assume that there exists two function $\beta \in \mathcal{F}$ such that for all $x, y \in X$*

$$m(Tx, Ty) \leq \beta(M(x, y))M(x, y), \tag{3.4}$$

where $M(x, y) = \max\{m(x, y), m(x, Tx), m(y, Ty)\}$. Then T has unique fixed point.

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