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GENERALIZATION OF SOME PROPERTIES OF C^* ALGEBRA TO BANACH ALGEBRAS

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Abstract. C^* Algebra is a special type of Banach algebra, which is a generalization of complex numbers, in C^* algebra, concepts such as positive elements, inequalities, and the like are defined based on properties, but a Banach algebra does not have the properties of C^* algebra. Some examples of Banach algebras like L^1 are not an C^* algebra. Now we want concepts such as positive elements, inequalities and the like to be defined in Banach algebras without relying on C^* algebra properties and these concepts are defined based on the characteristics of Banach algebra characters, therefore many of the results in C^* algebras can be generalized to Banach algebras. This article generalizes the definition of positive elements from C^* algebra to Banach algebra. Positive elements are defined based on characters (Multiplicative linear functional), Which contains almost all the properties of positive elements of C^* algebras, and most theorems are proved by this definition. By defining the inequalities in Banach algebra and studing the properties of the inequalities, valid results can be obtained.

1. Introduction

For a C^* algebra is positive element a from important concept and also C^* algebras are a from Specific concept algebra that it is special classis of

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banach algebra, positive element and also inequality is played a fundamental. inscription in C^* that is used in definition positive element from star in C^* . Some of the banach algebra are not C^* for example L^1 . In this article positive element expanded to banach algebra that can there in the make use of characters banach algebra . In this article some definition is based on the concept of positive element in banach algebra are presented. Initially extended a definition is presented for banach algebras. In this article characters are considered. Also compare the proposed definition with different versions from definition in C^* algebras. It is also possible to define inequalities in Banach algebra and to arrive at valid results by to be studied the properties of the inequalities.

Definition 1.1. If A is a banach algebra and $\mu(A)$ is not empty. then $\mu(A)$ is said separate points on A if for every nonzero element $a \in A$ there corresponds a $f \in \mu(A)$ such that $f(a) \neq 0$.

Definition 1.2. If A is a banach algebra and $\mu(A)$ is not empty, B banach subalgebra of A is said to be hereditary if for $a \in A^+$ and $b \in B^+$, the inequality $a \leq b$ implies $a \in B^+$.

Theorem 1.3. *If A is a banach algebra and $\mu(A)$ is not empty and $a, b, c \in A$. then the following hold*

1. *If a is hermitian then $0 \leq a^2$.*
2. *If a is hermitian and $n \in N$ is even then $0 \leq a^n$.*
3. *If a is hermitian and $0 \leq b$ then $0 \leq aba$.*

Proof. suppose $a \in \overline{A^+}$ hence there exists a sequence $\{a_n\} \subseteq A^+$ such that $a_n \xrightarrow{\|\|} a$ because $\{a_n\} \subseteq A^+$ therefore $f(a_n) \geq 0$ for all $f \in \mu(A)$ hence $f(a) \geq 0$ for all $f \in \mu(A)$, thus $a \in A^+$, so A^+ is closed in A. □

Theorem 1.4. *If A is a banach algebra and $\mu(A)$ is not empty. then $sp_A(a) = sp_{A^h}(a)$ for all $a \in A^h$.*

Proof. If $a \in A^h$ in A is invertible then there exists $b \in A$ such that $ab = 1$ therefore $f(a)f(b) = 1$ for all $f(a) \in \mu(A)$. since $a \in A^h$ then $f(a) \in R$ for all $f(a) \in \mu(A)$ hence $f(b) \in R$ for all $f(a) \in \mu(A)$. therefore $b \in A^h$ so $a \in A^h$ in A^h is invertible. □

Theorem 1.5. *If A is a unitary banach algebra and $\mu(A)$ is not empty and a is a hermitian element of A. then $\|a\|1 - a \in A^+$.*

Proof. Because $|f(a) \leq \|a\|$ for all $f \in \mu(A)$. and a is hermitian element of A. therefore $\|a\| - f(a) \geq 0$ for all $f \in \mu(A)$ so $\|a\|1 - a \in A^+$. □

Theorem 1.6. *If A is a unitary banach algebra and $\mu(A)$ is not empty. a is a hermitian element of A and $t \in R$. then $a \geq 0$ if $\|a - t\| \leq t$. in reverse direction, if $\|a\| \leq t$ and $a \geq 0$ then $\|a - t\| \leq t$.*

Proof. Suppose $\|a - t\| \leq t$. then $|f(a) - t| \leq \|a - t\| \leq t$ for all $f \in \mu(A)$. therefore a is a hermitian element of A and $a - t$ is hermitian. hence $-t \leq f(a) - t \leq t$ for all $f \in \mu(A)$ thus $0 \leq f(a)$ for all $f \in \mu(A)$ so $a \geq 0$.

now suppose $\|a\| \leq t$ and $a \geq 0$. then $0 \leq f(a) \leq t$ for all $f \in \mu(A)$. since $a \geq 0$ and $t \in R$ then $a - t$ is hermitian. hence $f(a) - t \in R$ for all $f \in \mu(A)$. so $|f(a) - t| \leq t$ for all $f \in \mu(A)$. we have $\|a - t\| = sup_{f \in A^*}|f(a) - t|$. on the other hand every extreme spectral state is a character. so $\|a - t\| = sup_{f \in \mu(A)}|f(a) - t|$ thus $\|a - t\| \leq t$. □

Lemma 1.7. *If A is a commutative banach algebra. then $a \in A$ is positive iff $Sp(a) \subseteq R^+$.*

Theorem 1.8. *If A is a commutative banach algebra and $a \in A$ is positive then there exists a,*
$b \in A$ such that $b \geq 0$ and $a = b^2$.

Proof. If $a = 0$ then theorem is clear.
If $a > 0$ because A is a commutative banach algebra, therefore $Sp(a) \subseteq [\ 0, +\infty)$. hence there exists a $\alpha \in A$ such that $a = e^{\alpha}$. now we consider $b = e^{\alpha/2}$ then $a = b^2$. □

Theorem 1.9. *If A is a commutative banach algebra and a, b are positive element of A, then inequality $a \leq b$ implies the inequality $a^{1/2} \leq b^{1/2}$.*

Proof. we show $a^2 \leq b^2 \Rightarrow a \leq b$. since $(a-b)(a+b) = a^2 - b^2$ then we have $(f(a) - f(b))(f(a) + f(b)) = f(a)^2 - f(b)^2$ for all $f \in \mu(A)$ so $a \leq b$. □

Theorem 1.10. *If A is a commutative banach algebra, $a \in A$ is positive and $n \in N$ then there exists a $b \in A$ such that $b \geq 0$ and $a = b^n$.*

Theorem 1.11. *If A is a commutative banach algebra, a, b are positive element of A and $n \in N$,then inequality $a \leq b$ implies the inequality $a^{\frac{1}{n}} \leq b^{\frac{1}{n}}$.*

Theorem 1.12. *If A is a banach algebra, $\mu(A)$ is not empty and $n \in N$. if $a \in A$ and $0 \leq a$ such that $\|a\| < 1$ then $a^n \leq a$.*

Proof. Since $\|a\| < 1$ then $f(a) \leq \|a\| < 1$ for all $f \in \mu(A)$. hence there exists $n \in N$ such that $f(a)^n \leq f(a)$ for all $f \in \mu(A)$. so $a^n \leq a$. □

Theorem 1.13. *If A is a commutative banach algebra, a, b are positive element of A and $\lambda \in [0, 1) \cap Q$,then inequality $a \leq b$ implies the inequality $a^{\lambda} \leq b^{\lambda}$.*

Lemma 1.14. *If A is a unitary banach algebra, and $\mu(A)$ is not empety. $S = \{a \in A | \|a\| \leq 1\}$ be it is (closed) unit ball. then $Int(S) \cap A^+ = \{a \in A^+ | \|a\| \leq 1\}$ is directed with respect to the partial order " $\geq$ " i.e for any $x,\ y \in Int(S) \cap A^+$ such that $z \geq x$ and $z \geq y$.*

Proof. Let

$$a = x(1-x)^{-1}\ ,\ b = y(1-y)^{-1}$$

$$z = (a+b)(1/2 + a + b)^{-1} = 1 - 1/2(1/2 + a + b)^{-1}$$

then $a,\ b,\ z \in A^+$ and $z \in Int(S) \cap A^+$.
the inequality is equivalent to the following inequality

$$1 - 1/2(1/2 + a + b)^{-1} \geq x \text{ or } 1 - x \geq 1/2(1/2 + a + b)^{-1}$$

the latter is equivalent to the

$$(1 - x)^{-1} \leq (1 + 2a + 2b)$$

it is clear that $(1 + 2a) \geq (1 - x)^{-1}$ by definition of a, thus $(1 + 2a + 2b) \geq (1 - x)^{-1}$. so $z \geq x$ similarly

$$z \geq y.$$

□

Theorem 1.15. *If A is a unitary commutative banach algebra. $S = \{a \in A \| \|a\| \leq 1\}$, if $x \in A^h$ and $\varepsilon > 0$, there is an element $a_0 \in Int(S) \cap A^+$ with the following property: if $a_0 \in Int(S) \cap A^+$ and $a > a_0$ then $\|x - xa\| < \varepsilon$.*

2. Example

Example 2.1. If A is a unitary commutative banach algebra and p is a projection of A. then pA is hereditary. for assuming $0 \leq b \leq pa$, then $0 \leq (1 - p)b \leq (1 - p)pa = 0$. so $(1 - p)b = 0$, therefore

$$b = pb \in pA.$$

Example 2.2. If $A = C(X)$ then $f \in C(X)$ is positive iff $f(x) \geq 0$, $\forall x \in X$ and $f \in C(X)$ is hermitian iff $f(x) \in R$, $\forall x \in X$.

Example 2.3. $L^1(R)$ with convolution from a commutative banach algebra. $f \in L^1(R)$ is positive iff $\widetilde{f}(\xi) \geq 0$ for $\forall \xi$. and $f \in L^1(R)$ is hermitian iff $\widetilde{f}(\xi) \in R$ for $\forall \xi$.

3. Main results

All the characters of inequlities and positive element are in C^* algebras is demonstrated as well as almost all characters is established here. Almost all the characters of inequlities and positive element that exist in C^* algebras can be expanded to banach algebras. Helbert C^* modules can be expanded and also Frames can be defined in banach algebra on based inequalities and may extended the other applications that its based on positive element and inequalities in C^* algebras.

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