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ON OPTIMALITY CONDITIONS FOR COMPOSITE UNCERTAIN MULTIOBJECTIVE OPTIMIZATION

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ABSTRACT. This article is concerned with a nonsmooth/nonconvex composite multiobjective optimization problem involving uncertain constraints in arbitrary Asplund spaces. We first establish necessary optimality conditions for weakly robust efficient solutions of the problem in terms of the limiting subdifferential. Then, sufficient conditions for the existence of (weakly) robust efficient solutions to such a problem are driven under the new concept of pseudo-quasi convexity for composite functions.

1. INTRODUCTION

Robust optimization approach considers the cases in which optimization problems often deal with uncertain data due to prediction errors, lack of information, fluctuations, or disturbances. Particularly, in most cases these problems depend on conflicting goals due to multiobjective decision makers which have different optimization criteria. So, the *robust multiobjective optimization* is highly interesting in optimization theory and important in applications. To the best of our knowledge, the most powerful results in this direction were established for finite-dimensional problems not dealing with composite functions. So, an infinite-dimensional framework would be proper to study when involving optimality and duality in composite optimization. From this, our main purpose in this paper is to investigate a

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nonsmooth/nonconvex multiobjective optimization problem with composition fields over arbitrary *Asplund* spaces.

Throughout this paper, we use standard notation of variational analysis; see, for example, [1]. Unless otherwise stated, all the spaces under consideration are *Asplund* with the norm $\|\cdot\|$ and the canonical pairing $\langle\cdot\,,\cdot\rangle$ between the space X in question and its *dual* X^* equipped with the *weak** *topology* w^*. For a given nonempty set $\Omega\subset X$, the symbols $\operatorname{co}\Omega$, $\operatorname{cl}\Omega$, and $\operatorname{int}\Omega$ indicate the *convex hull*, *topological closure*, and *topological interior* of Ω, respectively, while $\operatorname{cl}^*\Omega$ stands for the *weak** *topological closure* of $\Omega\subset X^*$. The *dual cone* of Ω is the set

$$\Omega^+ := \{x^*\in X^* \mid \langle x^*, x\rangle \geq 0,\ \ \forall x\in\Omega\}.$$

Besides, $\mathbb{R}^n_+$ signifies the nonnegative orthant of $\mathbb{R}^n$ for $n\in\mathbb{N}:=\{1,2,\dots\}$.

Suppose that $F: X\to W$ and $f: W\to Y$ be vector-valued functions between Asplund spaces, and that $K\subset Y$ be a pointed (i.e., $K\bigcap(-K)=\{0\}$) closed convex cone. We consider a *composite multiobjective optimization* problem:

$$\text{(CP)}\qquad \min\nolimits_K\ (f\circ F)(x)\qquad \text{s.t. } (g_i\circ G_i)(x)\leq 0,\quad i=1,2,\dots,n,$$

where the functions $G=(G_1,G_2,\dots,G_n): X\to Z$ and $g=(g_1,g_2,\dots,g_n): Z\to\mathbb{R}^n$ define the constraints on Asplund spaces. This problem in the face of data uncertainty in the constraints can be captured by the following *composite uncertain multiobjective optimization* problem:

$$\text{(CUP)}\qquad \min\nolimits_K\ (f\circ F)(x)\qquad \text{s.t. } (g_i\circ G_i)(x,v_i)\leq 0,\quad i=1,2,\dots,n,$$

where $x\in X$ is the vector of *decision* variable, v_i's are *uncertain* parameters and $v_i\in\mathcal{V}_i$ for some *sequentially compact topological* space $\mathcal{V}_i$, $\mathcal{V}:=\prod_{i=1}^{n}\mathcal{V}_i$, and $G_i: X\times\mathcal{V}_i\to Z\times\mathcal{U}_i$ and $g_i: Z\times\mathcal{U}_i\to\mathbb{R}$, $i=1,2,\dots,n$, are given functions for topological spaces $\mathcal{U}_i$, $\mathcal{U}:=\prod_{i=1}^{n}\mathcal{U}_i$.

For investigating the problem (CUP), we associate with it the so-called *robust* counterpart:

$$\text{(CRP)}\qquad \min\nolimits_K\ (f\circ F)(x)\qquad \text{s.t. } (g_i\circ G_i)(x,v_i)\leq 0,\quad \forall v_i\in\mathcal{V}_i,\ i=1,2,\dots,n.$$

A vector $x\in X$ is called a *robust feasible solution* of problem (CUP) if it is a *feasible solution* of problem (CRP). The *feasible set* C of problem (CRP) is defined by

$$C := \{x\in X \mid (g_i\circ G_i)(x,v_i)\leq 0,\ \forall v_i\in\mathcal{V}_i,\ i=1,2,\dots,n\}.$$

Definition 1.1. (i) We say that a vector $\bar{x}\in X$ is a *robust efficient solution* of problem (CUP), denoted by $\bar{x}\in\mathcal{S}$(CRP), if $\bar{x}$ is an *efficient solution* of problem (CRP), i.e., $\bar{x} \in C$ and

$$(f \circ F)(x) - (f \circ F)(\bar{x}) \notin -K \setminus \{0\}, \quad \forall x \in C.$$

(ii) A vector $\bar{x} \in X$ is called a *weakly robust efficient solution* of problem (CUP), denoted by $\bar{x} \in \mathcal{S}^w$(CRP), if $\bar{x}$ is a *weakly efficient solution* of problem (CRP), i.e., $\bar{x} \in C$ and

$$(f \circ F)(x) - (f \circ F)(\bar{x}) \notin -\mathrm{int}\, K, \quad \forall x \in C.$$

Motivated by the concept of pseudo-quasi generalized convexity in [4], we introduce a similar concept of pseudo-quasi convexity type for the compositions $f \circ F$ and $g \circ G$ to establish sufficient optimality conditions for (weakly) robust efficient solutions of problem (CUP).

Definition 1.2. (i) We say that $(f \circ F, g \circ G)$ is *type I pseudo convex* at $\bar{x} \in X$ if for any $x \in X$, $y^* \in K^+$, $w^* \in \partial\langle y^*, f\rangle(F(\bar{x}))$, $x^* \in \partial\langle w^*, F\rangle(\bar{x})$, $v_i^* \in \partial_x g_i(G_i(\bar{x}, v_i))$, and $x_i^* \in \partial_x\langle v_i^*, G_i\rangle(\bar{x}, v_i)$, $v_i \in \mathcal{V}_i(\bar{x})$, $i = 1, 2, \ldots, n$, there exists $\nu \in X$ such that

$$\langle y^*, f \circ F\rangle(x) < \langle y^*, f \circ F\rangle(\bar{x}) \Longrightarrow \langle x^*, \nu\rangle < 0,$$
$$(g_i \circ G_i)(x, v_i) \le (g_i \circ G_i)(\bar{x}, v_i) \Longrightarrow \langle x_i^*, \nu\rangle \le 0, \quad i = 1, 2, \ldots, n.$$

(ii) We say that $(f \circ F, g \circ G)$ is *type II pseudo convex* at $\bar{x} \in X$ if for any $x \in X \setminus \{\bar{x}\}$, $y^* \in K^+ \setminus \{0\}$, $w^* \in \partial\langle y^*, f\rangle(F(\bar{x}))$, $x^* \in \partial\langle w^*, F\rangle(\bar{x})$, $v_i^* \in \partial_x g_i(G_i(\bar{x}, v_i))$, and $x_i^* \in \partial_x\langle v_i^*, G_i\rangle(\bar{x}, v_i)$, $v_i \in \mathcal{V}_i(\bar{x})$, $i = 1, 2, \ldots, n$, there exists $\nu \in X$ such that

$$\langle y^*, f \circ F\rangle(x) \le \langle y^*, f \circ F\rangle(\bar{x}) \Longrightarrow \langle x^*, \nu\rangle < 0,$$
$$(g_i \circ G_i)(x, v_i) \le (g_i \circ G_i)(\bar{x}, v_i) \Longrightarrow \langle x_i^*, \nu\rangle \le 0, \quad i = 1, 2, \ldots, n.$$

Let $\Omega \subset X$ be *locally closed* around $\bar{x} \in \Omega$, i.e., there is a neighborhood U of $\bar{x}$ for which $\Omega \bigcap \mathrm{cl}\, U$ is closed. The *Fréchet normal cone* $\widehat{N}(\bar{x}; \Omega)$ and the *Mordukhovich normal cone* $N(\bar{x}; \Omega)$ to Ω at $\bar{x} \in \Omega$ are defined by

$$\widehat{N}(\bar{x}; \Omega) := \{x^* \in X^* \mid \limsup_{x \to \bar{x}} \frac{\langle x^*, x - \bar{x}\rangle}{\|x - \bar{x}\|} \le 0\},$$
$$N(\bar{x}; \Omega) := \operatorname*{Lim\,sup}_{x \to \bar{x}} \widehat{N}(x; \Omega),$$

where $x \xrightarrow{\Omega} \bar{x}$ stands for $x \to \bar{x}$ with $x \in \Omega$. If $\bar{x} \notin \Omega$, we put $\widehat{N}(\bar{x}; \Omega) = N(\bar{x}; \Omega) := \emptyset$.

For an extended real-valued function $\phi : X \to \overline{\mathbb{R}}$, the *limiting/Mordukhovich subdifferential* and the *regular/Fréchet subdifferential* of ϕ at $\bar{x} \in \mathrm{dom}\, \phi$ are given, respectively, by

$$\partial\phi(\bar{x}) := \{x^* \in X^* \mid (x^*, -1) \in N((\bar{x}, \phi(x)); \mathrm{epi}\, \phi)\}$$

and

$$\widehat{\partial}\phi(\bar{x}) := \{x^* \in X^* \mid (x^*, -1) \in \widehat{N}((\bar{x}, \phi(x)); \mathrm{epi}\, \phi)\}.$$

If $|\phi(\bar{x})| = \infty$, then one puts $\partial\phi(\bar{x}) = \widehat{\partial}\phi(\bar{x}) := \emptyset$.

Throughout this paper, we assume that the following assumptions hold:

Assumption 1.3. (See [2, p.131])

(A1) For a fixed $\bar{x} \in X$, F is locally Lipschitz at $\bar{x}$ and f is locally Lipschitz at $F(\bar{x})$.

(A2) For each $i = 1, 2, \ldots, n$, G_i is locally Lipschitz at $\bar{x}$ and uniformly on $\mathcal{V}_i$, and g_i is Lipschitz continuous on $G_i(\bar{x}, \mathcal{V}_i)$.

(A3) For each $i = 1, 2, \ldots, n$, the functions $v_i \in \mathcal{V}_i \mapsto G_i(\bar{x}, v_i) \in Z \times \mathcal{U}_i$ and $G_i(\bar{x}, v_i) \mapsto g_i(G_i(\bar{x}, v_i)) \in \mathbb{R}$ are locally Lipschitzian.

(A4) For each $i = 1, 2, \ldots, n$, we define real-valued functions ϕ_i and ϕ on X via

$$\phi_i(x) := \max_{v_i \in \mathcal{V}_i} (g_i \circ G_i)(x, v_i) \qquad \text{and} \qquad \phi(x) := \max_{i \in \{1,2,\ldots,n\}} \phi_i(x),$$

and we notice that above assumptions imply that ϕ_i is well defined on $\mathcal{V}_i$. In addition, ϕ_i and ϕ follow readily that are locally Lipschitz at $\bar{x}$, since each $(g_i \circ G_i)(\bar{x}, v_i)$ is (see [2, (H1), p.131] and [3, p.290]).

(A5) For each $i = 1, 2, \ldots, n$, the multifunction $(x, v_i) \in X \times \mathcal{V}_i \rightrightarrows \partial_x (g_i \circ G_i)(x, v_i) \subset X^*$ is weak* closed at $(\bar{x}, \bar{v}_i)$ for each $\bar{v}_i \in \mathcal{V}_i(\bar{x})$, where $\mathcal{V}_i(\bar{x}) = \{v_i \in \mathcal{V}_i \mid (g_i \circ G_i)(\bar{x}, v_i) = \phi_i(\bar{x})\}$.

In the rest of this section, we present a suitable constraint qualification in the sense of robustness, which is needed to get a so-called *robust Karush-Kuhn-Tucker* (KKT) *condition*.

Definition 1.4. (See [4, Definition 2.3]) Let $\bar{x} \in C$. We say that the *constraint qualification* (CQ) *condition* is satisfied at $\bar{x}$ if

$$0 \notin \mathrm{cl}^*\mathrm{co}(\cup\{\cup_{v_i^* \in \partial_x g_i(G_i(\bar{x}, v_i))} \partial_x \langle v_i^*, G_i \rangle(\bar{x}, v_i) \mid v_i \in \mathcal{V}_i(\bar{x})\}), \quad i \in I(\bar{x}),$$

where $I(\bar{x}) := \{i \in \{1, 2, \ldots, n\} \mid \phi_i(\bar{x}) = \phi(\bar{x})\}$.

It is worth to mention here that this condition (CQ) is reduced to the *extended Mangasarian-Fromovitz constraint qualification* (EMFCQ) in the *smooth* setting; see e.g., [1] for more details.

Definition 1.5. A point $\bar{x} \in C$ is said to satisfy the *robust* (KKT) *condition* if there exist $y^* \in K^+ \setminus \{0\}$, $\mu := (\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{R}^n_+$, and $\bar{v}_i \in \mathcal{V}_i$, $i = 1, 2, \ldots, n$, such that

$$0 \in \cup_{w^* \in \partial \langle y^*, f \rangle (F(\bar{x}))} \partial \langle w^*, F \rangle(\bar{x}) + \sum_{i=1}^n \mu_i \, \mathrm{cl}^*\mathrm{co}(\cup\{\cup_{v_i^* \in \partial_x g_i(G_i(\bar{x}, v_i))} \partial_x \langle v_i^*, G_i \rangle(\bar{x}, v_i) \mid v_i \in \mathcal{V}_i(\bar{x})\}),$$

$$\mu_i \max_{v_i \in \mathcal{V}_i} (g_i \circ G_i)(\bar{x}, v_i) = \mu_i \, (g_i \circ G_i)(\bar{x}, \bar{v}_i) = 0, \quad i = 1, 2, \ldots, n.$$

Therefore, the robust (KKT) condition defined above is guaranteed by the constraint qualification (CQ).

2. Robust necessary and sufficient optimality conditions

The first theorem establishes a necessary optimality condition in the sense of the limiting subdifferential for weakly robust efficient solutions of problem (CUP).

Theorem 2.1. *Suppose that* $\bar{x} \in \mathcal{S}^w$(CRP). *Then there exist* $y^* \in K^+$, $\mu := (\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{R}^n_+$, *with* $\|y^*\| + \|\mu\| = 1$, *and* $\bar{v}_i \in \mathcal{V}_i$, $i = 1, 2, \ldots, n$, *such that*

$$\begin{cases} 0 \in \cup_{w^* \in \partial \langle y^*, f\rangle (F(\bar{x}))} \partial \langle w^*, F\rangle(\bar{x}) + \sum_{i=1}^{n} \mu_i \, \mathrm{cl}^* \mathrm{co}(\cup \{\cup_{v_i^* \in \partial_x g_i(G_i(\bar{x}, v_i))} \partial_x \langle v_i^*, G_i\rangle(\bar{x}, v_i) \\ \qquad \mid \; v_i \in \mathcal{V}_i(\bar{x})\}), \\ \mu_i \max_{v_i \in \mathcal{V}_i} g_i(G_i(\bar{x}, v_i)) = \mu_i \, g_i(G_i(\bar{x}, \bar{v}_i)) = 0, \quad i = 1, 2, \ldots, n. \end{cases} \tag{2.1}$$

Furthermore, if the (CQ) *is satisfied at* $\bar{x}$, *then* (2.1) *holds with* $y^* \neq 0$.

Remark 2.2. Theorem 2.1 reduces to [4, Theorem 3.2] for the problem (UP), and [5, Proposition 3.9] and [2, Theorem 3.3] in the case of finite-dimensional multiobjective optimization. Note further that our approach here, which involves the fuzzy necessary optimality condition in the sense of the Fréchet subdifferential and the inclusion formula for the limiting subdifferential of maximum functions in the setting of Asplund spaces, is totally different from the last two presented in the aforementioned papers.

The forthcoming theorem presents a (KKT) sufficient optimality conditions for (weakly) robust efficient solutions of problem (CUP).

Theorem 2.3. *Assume that* $\bar{x} \in C$ *satisfies the robust* (KKT) *condition.*

(i) *If* $(f \circ F, g \circ G)$ *is type I pseudo convex at* $\bar{x}$, *then* $\bar{x} \in \mathcal{S}^w$(CRP).

(ii) *If* $(f \circ F, g \circ G)$ *is type II pseudo convex at* $\bar{x}$, *then* $\bar{x} \in \mathcal{S}$(CRP).

Remark 2.4. Theorem 2.3 reduces to [4, Theorem 3.4] and [5, Theorem 3.10], and develops [2, Theorem 3.11] and [6, Theorem 3.2] under pseudo-quasi convexity assumptions.

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