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APPLICATIONS OF TENSOR ANALYSIS TO COMPUTE THE CURVATURE AND TORSION FOR IMPLICIT CURVES

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ABSTRACT. In the this paper, curvature and torsion formulas will be computed for an implicit curve in $(n+1)$-dimension by using tensor analysis and operations. Then, Goldman's results for computing the torsion of an implicit curve have been extended in $\mathbb{R}^{n+1}$ Euclidean space. In addition, some useful formulas to calculate the higher order analogues of the torsion in $(n+1)$-dimensions will be derived in this paper, using tensor operations.

1. Introduction

Curvature formulas of surfaces and curves in Euclidean space have been developed by many mathematicians so far by using differential geometry. The differential geometry of curves and surfaces can be found in textbook such as in Spivak(1975) and Stocker(1969).T. Maekawa and N.H. Patrikalakis (2001) presented Ferenet-Serret formulaes for space curves. Also they spoke about principal curvatures of explicit surface. Bajaj and Kim (1991) and B. Linn (1997) presented a formula to compute the curvature for an implicit plannar curve. R. Osserman considered some relations between sectional curvatures and the scalar curvature in (n)-dimensional Euclidean

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space. Klingenberg (1978) provided a curvature formula for curves which are intersections of some equations in $\mathbb{R}^3$, $\mathbb{R}^4$ and $\mathbb{R}^n$. K. Nomizu worked on certain conditions to drive the tensor of curvature for hypersurfaces. Curvature formulas to calculate mean and Gaussian curvatures for arbitrary surfaces provided by Turkiyyah(1997) and Belyaev(1998). P. Hartman and L. Nirenberg considered some no change properties of hypersurfaces of dimension n immersed in $(n+1)$-dimensional Euclidean space. Different formulas to calculate the curvature of intersection curves in (3)-dimensional Euclidean space by using implicit function theorem were given by Hartmann(1996).

H. Schlichtkrull (2011) provided some formulas to calculate geodesic and normal curvatures for an arbitrary curve and relation between components of Reimann curvature tensor and the second fundamental form of implicit and explicit surfaces. Osherand Fedkiv(2003) computed some formulas to calculate the curvature for implicit curves and surfaces by using Level set method. R. Goldman (2005) found formulas to compute the curvature of curves in $(n+1)$-dimensions which were intersections of (n) hypersurfaces but for the torsion of curves, only a formula in $\mathbb{R}^3$ was driven. Formulas to calculate first, second and third curvatures of intersection curves in $\mathbb{R}^4$ were provided with O. Alessio (2009) by using implicit function theorem. X. Ye and T. Maekawa. Mohamed. S. Lone and O.Alessio and M. H. Shahid (2016) used some formulas to compute $\kappa_1, \kappa_2, \kappa_3, \kappa_4$ and geodesic curvature in $\mathbb{R}^5$ but formula for higher order analogues of κ_4 was not provided. The study of curvature, torsion and higher-order analogous for implicit curves (for example see [1, 2]).

2. Main Results

It is well known from elementary geometry that a curve in $\mathbb{R}^3$ can be described by $x = x(t)$, $y = y(t)$ and $z = z(t)$. $\quad (t_1 < t < t_2)$

The purpose of this work is to provide the curvature formula for an implicit curve in $(n+1)$-dimensions which is generated by the intersection of n implicit simultaneous equations[4].

A parameterized continuous curve in $\mathbb{R}^3$ is a continuous map $\gamma : I \to \mathbb{R}^3$, where $I \subseteq R$ is an open interval (of end points $0 < a < b < \infty$). The parametric curve is assumed to be of class 3. The implicit representation for a space curve can be expressed as intersection curve between two implicit surfaces $F(x, y, z) = 0$ and $G(x, y, z) = 0$.

If the two implicit equations $F = 0$ and $G = 0$ can be solved for two of the variables in terms of the third, for example $\dot{y}$ and $\dot{z}$ in terms of $\dot{x}$, we obtain the curvature formula.This is always possible at least locally when $\dot{x}$ is not equal to zero.

Let us consider two implicit simultaneous equations which intersect each other in an arbitrary curve which lies in 3-dimensional Euclidean space. We can take first and second differential from two implicit functions to drive $\dot{y}$

, $\dot{z}$, $\ddot{y}$ and $\ddot{z}$:

$$\begin{cases} F(x,y,z) = z - f(x) = 0 \\ G(x,y,z) = z - g(y) = 0 \end{cases} \Rightarrow \begin{cases} \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} = -\dot{x}f_x + \dot{z} = 0 \\ \\ \frac{\partial G}{\partial x}\frac{dx}{dt} + \frac{\partial G}{\partial y}\frac{dy}{dt} + \frac{\partial G}{\partial z}\frac{dz}{dt} = -\dot{y}g_y + \dot{z} = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \dot{y} = \dot{x}\frac{f_x}{g_y} \\ \dot{z} = \dot{x}f_x \end{cases} \quad \Rightarrow \begin{cases} \ddot{y} = \ddot{x}\frac{f_x}{g_y} + \dot{x}^2(\frac{g_y^2 f_{xx} - f_x^2 g_{yy}}{g_y^3}) \\ \ddot{z} = \ddot{x}f_x + \dot{x}^2 f_{xx} \end{cases}$$

The curvature formula for the parametric curve γ is

$$\kappa = \frac{\|(\dot{y}\ddot{z} - \dot{z}\ddot{y})\widehat{e_x} + (\dot{z}\ddot{x} - \dot{x}\ddot{z})\widehat{e_y} + (\dot{x}\ddot{y} - \dot{y}\ddot{x})\widehat{e_z}\|}{\|\dot{x}\widehat{e_x} + \dot{y}\widehat{e_y} + \dot{z}\widehat{e_z}\|^3}$$

Gradients for implicit functions F and G are given by

$$\begin{cases} \overrightarrow{\nabla}F = \overrightarrow{P_1} = F_x\widehat{e_x} + F_y\widehat{e_y} + F_z\widehat{e_z} = -f_x\widehat{e_x} + \widehat{e_z} \\ \overrightarrow{\nabla}G = \overrightarrow{P_2} = G_x\widehat{e_x} + G_y\widehat{e_y} + G_z\widehat{e_z} = -g_y\widehat{e_y} + \widehat{e_z} \end{cases}$$

Now we compute the cross product of vectors $\overrightarrow{P_1}$ and $\overrightarrow{P_2}$:

$$\overrightarrow{u} = \overrightarrow{P_1} \times \overrightarrow{P_2} = g_y\widehat{e_x} + f_x\widehat{e_y} + f_x g_y\widehat{e_z}$$

So we can proof this formula for the curvature:

$$\kappa = \frac{\left\|\overrightarrow{u}.(\overrightarrow{\nabla}\mathbf{B}).\overrightarrow{u}\right\|}{\left\|\overrightarrow{u}\right\|^3} = \frac{\left\|(\overrightarrow{\nabla}F \times \overrightarrow{\nabla}G).(\overrightarrow{\nabla}\mathbf{B}).(\overrightarrow{\nabla}F \times \overrightarrow{\nabla}G)\right\|}{\left\|\overrightarrow{\nabla}F \times \overrightarrow{\nabla}G\right\|^3} \tag{2.1}$$

Now we introduce two new characters $\lambda_1 = \overrightarrow{u}.\mathbf{T}_1.\overrightarrow{u}$ and $\lambda_2 = \overrightarrow{u}.\mathbf{T}_2.\overrightarrow{u}$:
$\overrightarrow{u}.(\mathbf{T}_2 \bigotimes \overrightarrow{P_1} - \mathbf{T}_1 \bigotimes \overrightarrow{P_2}).\overrightarrow{u} = \lambda_2\overrightarrow{P_1} - \lambda_1\overrightarrow{P_2}$

And hence

$$\kappa = \frac{\left\|\lambda_2\overrightarrow{P_1} - \lambda_1\overrightarrow{P_2}\right\|}{\left\|\overrightarrow{u}\right\|^3} \tag{2.2}$$

After using above formulas we have

$$\left\|\overrightarrow{u}\right\| = \sqrt{\Omega_{i'j'k'l'}P_1^{i'}P_2^{j'}P_1^{k'}P_2^{l'}}, \tag{2.3}$$

$$\Omega_{i'j'k'l'} = \begin{vmatrix} \delta_{i'k'} & \delta_{i'l'} \\ \delta_{j'k'} & \delta_{j'l'} \end{vmatrix}$$

$$\Omega_{ijkl} = \begin{vmatrix} \delta_{ik} & \delta_{il} \\ \delta_{jk} & \delta_{jl} \end{vmatrix}$$

And the tensor form of the curvature may be written as

$$\kappa = \frac{\sqrt{\Omega_{ijkl}\lambda_i\lambda_k(\overrightarrow{P_j}.\overrightarrow{P_l})}}{\left\{\Omega_{i'j'k'l'}P_1^{i'}P_2^{j'}P_1^{k'}P_2^{l'}\right\}^{\frac{3}{2}}} \tag{2.4}$$

$$\lambda_b = \overrightarrow{u}.(\overrightarrow{\nabla}\overrightarrow{P_b}).\overrightarrow{u} = \varepsilon_{\alpha\beta\tau}\varepsilon_{\eta\sigma\omega}P_1^{\alpha}P_2^{\beta}P_1^{\eta}P_2^{\sigma}P_{b,\tau}^{\omega}$$

For all $i, j, k, l = 1, 2,$ $\quad i', j', k', l', \alpha, \beta, \tau, \eta, \sigma, \omega = 1, 2, 3$ and $b = i, k \in \{1, 2\}$.
It is interesting to extend (2.2) for implicit space curves to a formula for implicit curves in $(n+1)$-dimensions that is, to curves is generated by the intersection of n hyper surfaces which lying in $\mathbb{R}^{n+1}(n = 2, 3, \cdots)$.

3. Conclusions

In this work, Curvature and torsion formulas for parametric planar and space curves are derived in differential geometry. Due to the application of curve geometry in the analysis of space-time, geometric quantities in higher dimensions have been studied. Since driving closed formulas for the curvature and the torsion and also higher-order analogues of the torsion for implicit surfaces defined by the intersection of implicit equations $F_1(x_1, \cdots, x_{n+1}) = 0 \cap \cdots \cap F_n(x_1, \cdots, x_{n+1}) = 0$ leads to complicated formulas, studying the geometric quantities for implicit curves and surfaces is the main focus of many researchers. Closed formulas for the curvature in $(n+1)$-dimensions and for the torsion in 3-dimensions for implicit curves have been derived by Ron. Goldman in [3]. Finally, by using the MATLAB program to calculate the geometric values of well-known implicit curves, the obtained formulas are verified.

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