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ESSENTIAL NORM OF SUBSTITUTION INTEGRAL OPERATORS ON ORLICZ SPACES

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Abstract. In this paper, we determine the lower and upper estimates for the essential norm of a substitution vector-valued integral operators on Orlicz spaces under certain conditions.

1. Introduction

Let $\theta : \mathbb{R} \to \mathbb{R}^+$ be a continuous convex function such that
(1) $\theta(x) = 0$ if and only if $x = 0$.
(2) $\lim_{x\to\infty} \theta(x) = \infty$.
(3) $\lim_{x\to\infty} \frac{\theta(x)}{x} = \infty$.
The convex function θ is called Young's function. With each Young's function θ, one can associate another convex function $\theta^* : \mathbb{R} \to \mathbb{R}^+$ having similar properties, which is defined by

$$\theta^*(y) = \sup\{x|y| - \theta(x) : x \geq 0\}.$$

The convex function θ^* is called complementary Young function to θ. Let $X = (X, \Sigma, \mu)$ be a σ-finite complete measure space. All comparisons between two functions or two sets are to be interpreted as holding up to a μ-null set. If θ is a Young function, then the set of Σ-measurable functions

$$L^\theta(\mu) = \{f : X \to \mathbb{C} : \exists\alpha > 0, \int_X \theta(\alpha|f|)d\mu < \infty\}$$

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is a Banach space, with respect with the Luxemburg norm defined by

$$\|f\|_\theta = \inf\{\delta > 0 : \int_X \theta(\frac{|f|}{\delta})d\mu \leq 1\}.$$

$(L^\theta(\mu), \|.\|_\theta)$ is called Orlicz space. A Young function θ is said to satisfy the Δ_2-condition(globally) if $\theta(2x) \leq k\theta(x), x > x_0(x_0 = 0)$, for some constant$k > 0$. For more details on Orlicz spaces, we refer to [3, 4, 5].

Let $\varphi : X \to X$ be a non-singular measurable transformation; i.e. $\mu \circ \varphi^{-1} \ll \mu$. It is assumed that the Radon-Nikodym derivative $h = d\mu\circ\varphi^{-1}/d\mu$ is almost everywhere finite-valued, or equivalently $\varphi^{-1}(\Sigma) \subseteq \Sigma$ is a sub-σ-finite algebra [6]. We have the following change of variable formula:

$$\int_{\varphi^{-1}(A)} f \circ \varphi d\mu = \int_A hfd\mu \qquad A \in \Sigma, f \in L^0(\Sigma).$$

Any nonsingular measurable transformation φ induces a linear operator (composition operator) C_φ from $L^0(\mu)$ into itself defined by

$$C_\varphi(f)(x) = f(\varphi(x)) \quad ; x \in X, \quad f \in L^0(\mu),$$

where $L^0(\Sigma)$ denotes the linear space of all equivalence classes of Σ-measurable functions on X. Here non-singularity of φ guarantees that the operator C_φ is well defined as a mapping from $L^0(\Sigma)$ into itself. If C_φ maps on Orlicz space $L^\theta(\mu)$ into itself , then C_φ is called composition operator on $L^\theta(\mu)$. For a given complex Hilbert space $\mathcal{H}$, let $u : X \to \mathcal{H}$ be a mapping. We say that u is weakly measurable if for each $g \in \mathcal{H}$ the mapping $x \mapsto \langle u(x), g\rangle$ of X to $\mathbb{C}$ is measurable. We will denote this map by $\langle u, g\rangle$.

Definition 1.1. Let $\varphi : X \to X$ be a non-singular measurable transformation and $\mathcal{C}_\varphi$ be a composition operator on $L^\theta(X)$. Also let $u : X \to \mathcal{H}$ be a weakly measurable function. Then the pair (u, φ) induces a substitution vector-valued integral operator $T_u^\varphi : L^\theta(\mu) \to \mathcal{H}$ defined by

$$\langle T_u^\varphi f, g\rangle = \int_X \langle u, g\rangle f \circ \varphi d\mu, \quad f \in L^\theta(\mu).$$

It is easy to see that T_u^φ is well defined and linear.

For a sub-σ-finite algebra $\mathcal{A} \subseteq \Sigma$, the conditional expectation operator associated with $\mathcal{A}$ is the mapping $f \to E^{\mathcal{A}}f$, defined for all non-negative f as well as for all $f \in L^p(\Sigma), 1 \leq p \leq \infty$, where $E^{\mathcal{A}}f$, by Radon-Nikodym Theorem, is the unique $\mathcal{A}$-measurable function satisfying

$$\int_A fd\mu = \int_A E^{\mathcal{A}}fd\mu, \quad \forall A \in \mathcal{A}.$$

For more details on the properties of $E^{\mathcal{A}}$ see [2].

Now we are going to investigate closed-range substitution vector-valued integral operators on Orlicz spaces. Next, we determine the essential norm these type operators.

First, we characterize the closedness of range of a substitution vector-valued integral operator from $L^{\theta}(\mu)$ to $\mathcal{H}$.
We start by the following lemma. Put $J := \cup_{\lambda\in\mathcal{H}_1}\sigma(hE(|\langle u,\lambda\rangle|\circ\varphi^{-1})$.

Lemma 1.2. *Let T_u^{φ} be a bounded substitution vector-valued integral operator from $L^{\theta}(\mu)$ to $\mathcal{H}$ and there is a constant $c>0$ such that $\sup_{\lambda\in\mathcal{H}_1} hE(|\langle u,\lambda\rangle|\circ \varphi^{-1}\geq c$ on J, then $T_u^{\varphi}\mid_J$ is injective..*

Theorem 1.3. *Let T_u^{φ} be a bounded substitution vector-valued integral operator from $L^{\theta}(\mu)$ to $\mathcal{H}$. Then, the following statements are hold.*

(i) Suppose T_u^{φ} from $L^{\theta}(\mu)$ to $\mathcal{H}$ has closed range and $x\prec\theta$ then there is a constant $c>0$ such that $\sup_{\lambda\in\mathcal{H}_1} hE(|\langle u,\lambda\rangle|\circ\varphi^{-1}\geq c$ on J.

(ii) If there is a constant $c>0$ such that $\sup_{\lambda\in\mathcal{H}_1} hE(|\langle u,\lambda\rangle|\circ\varphi^{-1}\geq c$ on J and $\theta\prec x$, then T_u^{φ} from $L^{\theta}(\mu)$ to $\mathcal{H}$ has closed range

In following, we give equivalent conditions with conditions of Theorem 3.2.

Lemma 1.4. *Let $\mathcal{B}$ be the collection of all Σ-measurable sets E such that*

(i) $\mu(E)<\infty$ and

(ii) whenever $F\in\Sigma$ satisfies $F\subseteq E$ and $\sup_{\lambda\in\mathcal{H}_1}\int_{\varphi^{-1}(F)}|\langle u,\lambda\rangle|d\mu=0$, then $\mu(F)=0$.
Suppose that $E\in\Sigma$ and $\mu(E)<\infty$. Hence, $E\in\mathcal{B}$ if and only if $E\subseteq J$.

Proposition 1.5. *The following statements are equivalent.*

(i) There is a constant $c>0$ such that $\sup_{\lambda\in\mathcal{H}_1} hE(|\langle u,\lambda\rangle|)\circ\varphi^{-1}\geq c$ on J.

(ii) There is a constant $\alpha>0$ such that $\sup_{\lambda\in\mathcal{H}_1}\int_{\varphi^{-1}(E)}|\langle u,\lambda\rangle|d\mu\geq\alpha\mu(E)$ for all $E\in\mathcal{B}$

Let $\mathcal{B}$ be a Banach space and $\mathcal{K}$ be the set of all compact operators on $\mathcal{B}$, the essential norm of T means the distance from T to $\mathcal{K}$ in the operator norm, namely

$$\|T\|_e=\inf\{\|T-S\|: S\in\mathcal{K}\}.$$

Clearly, T is compact if and only if $\|T\|_e=0$. As is seen [**?**], the essential norm plays an interesting role in the compact problem of concrete operators.

Theorem 1.6. *Let T_u^{φ} be a bounded operator on $L^{\theta}(\mu)$. Also let $\alpha=inf\{r>0, N_r\,\text{consists of finitely many atoms}\}$ and*

$$N_r=\{x\in X:=\sup_{\lambda\in\mathcal{H}_1}\|hE(|\langle u,\lambda\rangle|)\circ\varphi^{-1}\geq r\}.$$

Then we obtain that
(i) $\|T_u^{\varphi}\|_e = 0$ *if and only if* $\alpha = 0$.
(ii) $\|T_u^{\varphi}\|_e \geq \frac{1}{a}\alpha$ *where* $\theta \prec x$*(i.e. for some* $a > 0$ *we have* $\theta(x) \leq ax$*. In particular if* $a \leq 1$ *we have* $\|T_u^{\varphi}\|_e \geq \alpha$
(iii) $\|T_u^{\varphi}\|_e \leq a\alpha$ *where* $x \prec \theta$*(i.e. for some* $a > 0$ *we have* $\theta(x) \leq ax$*.In particular if* $a \leq 1$ *we have* $\|T_u^{\varphi}\|_e \leq \alpha$

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