EFFECTIVE IMPLEMENTATION OF LEGENDRE POLYNOMIALS IN PRICING DISCRETELY MONITORED DOUBLE BARRIER OPTION

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Abstract. In the present paper, Legendre polynomials are effectively implemented in pricing discrete double barrier options which are commonly done through recursive solving Black-Scholes PDEs in the monitoring intervals. By using orthogonal projection based on Legendre polynomials, we could obtain an operational matrix to approximate the price of the option.

1. Introduction

A knock-out double barrier option is an option that is deactivated when the price of the underlying asset touches each of the two predetermined barriers before the expiry date at monitoring dates. Various approaches have been proposed for pricing barrier options. An analytical method is derived by Fusai et al. in [1] based on z-transform. The method of finite element is used by Golbabai et al.[2]. Milev and Tagliani presented a numerical algorithm for pricing discrete double barrier options [3]. Farnoosh et al. [4, 5] provide methods for pricing discretely monitored (single or double) barrier options that work even for the case of time-dependent parameters. In this paper, Legendre Polynomials is effectively implemented as an orthogonal basis for the projection method that causes to operational matrix form. Computational time is almost fixed and not affected by the number of monitoring dates. According to the Black-Scholes framework, the price of

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discretely monitored double barrier call option as a function of stock price s at time $t \in (t_m, t_{m+1})$, namely $\mathcal{C}(s,t,m)$, is obtained from forward solving the following partial differential equations with the initial conditions[6]:

$$-\frac{\partial \mathcal{C}}{\partial t} + \mu s \frac{\partial \mathcal{C}}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 \mathcal{C}}{\partial s^2} - r\mathcal{C} = 0, \tag{1.1}$$

$$\mathcal{C}(s, t_0, 0) = (s - E)\,\mathbf{1}_{(\max(E,L) \le s \le U)}\,,$$

$$\mathcal{C}(s, t_m, m) = \mathcal{C}(s, t_m, m-1)\,\mathbf{1}_{(L \le s \le U)};\ m = 1, 2, ..., M-1\ .$$

The constant coefficients μ and σ are risk-free rate and volatility respectively. Also, the constants E, L, and U are exercise price, lower and upper barrier respectively. In the following, two changes of variables are performed. At first, the function $P(z,t,m)$ is defined as $P(z,t,m) := \mathcal{C}(s,t,m)$ where $z = \ln\left(\frac{s}{L}\right)$, $E^* = \ln\left(\frac{E}{L}\right)$, $\mu^* = \mu - \frac{\sigma^2}{2}$, $U^* = \ln\left(\frac{U}{L}\right)$, $\delta = \max\{E^*, 0\}$. Then the partial differential equation (1.1) and its initial conditions are changed into:

$$-P_t + \mu^* P_z + \frac{\sigma^2}{2} P_{zz} - \mu P = 0, \tag{1.2}$$

$$P(z, t_0, 0) = L\left(e^z - e^{E^*}\right)\mathbf{1}_{(\delta \le z \le U^*)},$$

$$P(z, t_m, m) = P(z, t_m, m-1)\,\mathbf{1}_{(0 \le z \le U^*)}\,;\ m = 1, 2, ..., M-1\ .$$

As a second step, the following transformation is applied:

$$P(z, t_m, m) = e^{\alpha z + \beta t} g(z,t,m),$$

where $\alpha = -\frac{\mu^*}{\sigma^2}$; $c^2 = \frac{\sigma^2}{2}$; $\beta = \alpha\mu^* + \alpha^2\frac{\sigma^2}{2} - \mu$.
Therefore, the partial differential equation (1.2) and its initial conditions are led to:

$$-g_t + c^2 g_{zz} = 0, \tag{1.3}$$

$$g(z, t_0, 0) = Le^{-\alpha z}\left(e^z - e^{E^*}\right)\mathbf{1}_{(\delta \le z \le U^*)},$$

$$g(z, t_m, m) = g(z, t_m, m-1)\,\mathbf{1}_{(0 \le z \le U^*)};\ m = 1, ..., M-1.$$

The resulting expressions in (1.3) are known as heat equations. Analytical solutions to the heat equations at the monitoring dates of equal distances $\tau = \frac{T}{M}$ or equivalently $t_m = m\tau$, are denoted by $f_m(z) := g(z, t_m, m-1)$ and evaluated as follows, see e.g [7];

$$f_0(z) = Le^{-\alpha z}\left(e^z - e^{E^*}\right)\mathbf{1}_{(\delta \le z \le U^*)}, \tag{1.4}$$

$$f_m(z) = \mathcal{K}(f_{m-1}(z)),\ m = 2, 3, ..., M-1, \tag{1.5}$$

where the compact operator $\mathcal{K} : L^2([0, U^*]) \to L^2([0, U^*])$ is defined as follows:

$$\mathcal{K}(f)(z) := \int_0^{U^*} \frac{1}{\sqrt{4\pi c^2 \tau}} e^{-\frac{(z-\xi)^2}{4c^2\tau}} f(\xi) d\xi. \tag{1.6}$$

According to the above stages, the price of the knock-out discrete double barrier European call Option at expiry date T is evaluated by the following formula:

$$\mathcal{C}(s_0, T, M-1) = e^{(\alpha z_0 + \beta T)} f_{M-1}(z_0), \tag{1.7}$$

where $z_0 = \ln\left(\frac{s_0}{L}\right)$.

2. Implementation of Legendre Polynomials

$$p_i(x) = xp_{i-1}(x) + \left(\frac{i}{i+1}\right)(xp_{i-1}(x) - p_{i-2}(x)),$$

where $p_0(x) = 1$, and $p_1(x) = x$. The $\{p_i(x)\}_{i=0}^{\infty}$ is an orthogonal basis for $L^2[-1,1]$. Now, we define $\tilde{p}_i(x) := \sqrt{\frac{2i+1}{U^*}}p_i\left(\frac{U^*}{2}x + \frac{U^*}{2}\right)$. These functions constitute an orthonormal basis for $L^2[0,U^*]$. Consider $\Pi_n = span\,\{\tilde{p}_i(x)\}_{i=0}^{n}$ be the space of all polynomials with degrees less than or equal to n and also $P_n : L^2[0,U^*] \to \Pi_n$ be orthogonal projection operator, that is defined as follows:

$$\forall f \in L^2[0,U^*] \quad P_n(f) = \sum_{i=0}^{n} \langle f, \tilde{p}_i(x)\rangle \tilde{p}_i(x), \tag{2.1}$$

where $\langle\ .\ ,\ .\ \rangle$ indicates the usual inner product.

Now, we define $\tilde{f}_{m,n} = P_n\mathcal{K}\left(\tilde{f}_{m-1,n}\right) = (P_n\mathcal{K})^m(f_0)$, $m \geq 2$ where $(P_n\mathcal{K})(f) = P_n(\mathcal{K}(f))$. Since the continuous projection operators P_n converge pointwise to identity operator I, then operator $P_n\mathcal{K}$ is also a compact operator and it could be shown that

$$\lim_{n\to\infty} \|(P_n\mathcal{K})^m - \mathcal{K}^m\| = 0. \tag{2.2}$$

Since, $\tilde{f}_{m,n} \in \Pi_n$ for $m \geq 1$, we can write

$$\tilde{f}_{m,n} = \sum_{i=0}^{n} a_{mi}\tilde{p}_i(z) = \Phi_n(x)F_m,$$

where $F_m = [a_{m0}, a_{m1}, \cdots, a_{m2^j}]'$ and $\Phi_n = [\tilde{p}_0(x), \tilde{p}_1(x), \cdots, \tilde{p}_n(x)]'$. So we obtain

$$\tilde{f}_{m,n} = (P_n\mathcal{K})^{m-1}\left(\tilde{f}_{1,n}\right). \tag{2.3}$$

Because Π_n is a finite-dimensional linear space, so the linear operator $P_n\mathcal{K}$ on Π_n could be considered as a $(n+1)\times(n+1)$ matrix K. Consequently, equation 2.3 can be written as the following matrix operator form:

$$\tilde{f}_{m,n} = \Phi_n' K^{m-1} F_1. \tag{2.4}$$

For computation of the option price by 2.4, it is enough to calculate the matrix operator K and the vector F_1:

$$F_1 = [a_{10}, a_{11}, \cdots, a_{1n}]', \quad K = (k_{ij})_{(n+1)\times(n+1)},$$

$$a_{1i} = \int_0^{U^*}\int_\delta^{U^*} \tilde{p}_i(\eta)\kappa(\eta-\xi,\tau)f_0(\xi)d\xi d\eta\ ,\ 0 \leq i \leq n,$$

$$k_{ij} = \int_0^{U^*}\int_0^{U^*} \tilde{p}_i(\eta)\tilde{p}_j(\xi)\kappa(\eta-\xi,\tau)d\xi d\eta\ ,$$

where $\kappa(z,t) = \frac{1}{\sqrt{4\pi c^2 t}} e^{-\frac{z^2}{4c^2 t}}$. The matrix form of relation 2.4 implies that the computational time of the presented algorithm be nearly fixed when monitoring dates increase. The complexity of our algorithm is $\mathcal{O}(n^2)$ that does not depend on the number of monitoring dates.

3. Numerical Result

Here, price of a double knock-out barrier option with $T = 0.5$, $\mu = 0.05$, $\sigma = 0.25$, $s_0 = 100$ $E = 100$, $U = 120$ and different level of lower barrier L is approximated by presented method. The numerical results are reported and compared with some other ones. The CPU time of the Presented method does not increase significantly when the number of monitoring dates increases.

M	L	Legendre($n = 16$)	Quad-K30	AMM-8	Benchmark
	80	2.4499	2.4499	2.4499	2.4499
	90	2.2028	2.2028	2.2027	2.2028
5	95	1.6831	1.6831	1.6830	1.6831
	99	1.0811	1.0811	1.0811	1.0811
	99.9	0.9432	0.9432	0.9433	0.9432
CPU		0.52 s			
	80	1.9420	1.9420	1.9419	1.9420
	90	1.5354	1.5354	1.5353	1.5354
25	95	0.8668	0.8668	0.8668	0.8668
	99	0.2931	0.2931	0.2932	0.2931
	99.9	0.2023	0.2023	0.2024	0.2023
CPU		0.54 s			

Table 1: Double knock-out barrier option: $T = 0.5$, $\mu = 0.05$, $\sigma = 0.25$, $s_0 = 100$, $E = 100$.

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