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ON THE DUALITY OF CONTROLLED G-FRAMES OF OPERATORS

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Abstract. In this paper we introduce the concept of duals of a controlled g-frame of operators and then, some conditions under which a controlled K-g-Bessel sequence is a controlled dual frame of a given controlled K-g-frame is presented. In the sequel, we discuss the structure of the canonical dual of a controlled K-g-frame and some other related results.

1. Introduction

Weighted and controlled frames have been recently introduced to improve the numerical efficiency of iterative algorithms for inverting the frame operator on abstract Hilbert spaces. Afterwards, this topic has been generalized for g-frames, fusion frames and K-frames. Controlled K-g frames, as one of the newest generalizations of frames, introduced in [3], are obtained from the combination of controlled frames, K-frames and g-frames. Since the frame operator of a controlled K-g-frame may not be invertible in general, there is no classical canonical dual for a controlled K-g-frame. So, it is interesting to find or even define the canonical dual of a controlled K-g-frame. In this paper, we propose the concept of dual and canonical dual of controlled K-g-frames from the operator theoretic point of view. Moreover, we give several equivalent characterizations of them. Throughout the paper, $\mathcal{H}$ is

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a separable Hilbert space and $\{\mathcal{H}_i : i \in \mathbb{I}\}$ is a sequence of Hilbert spaces, where $\mathbb{I}$ is an at most countable index set. We denote by $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ the set of all bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$. For $U \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, we use the notations U^*, $R(U)$ and $N(U)$ to denote respectively the adjoint operator, the range and the null space of U. We define $GL(\mathcal{H}_1, \mathcal{H}_2)$ as the set of all bounded linear operators with a bounded inverse, and similarly for $GL(\mathcal{H})$. A bounded operator T is called positive (respectively non-negative), if $\langle Tf, f\rangle > 0$, for all $0 \neq f \in \mathcal{H}$, (respectively $\langle Tf, f\rangle \geq 0$, for all $f \in \mathcal{H}$). The set of all positive operators in $GL(\mathcal{H})$ will be denoted by $GL^+(\mathcal{H})$. Notice that $U \in GL^+(\mathcal{H})$ if and only if U is positive and $U^{\frac{1}{2}} \in GL(\mathcal{H})$.

Definition 1.1. [3] Suppose that $K \in \mathcal{B}(\mathcal{H})$ and $C, C' \in GL^+(\mathcal{H})$, which commute with each other. The family $\{\Lambda_i \in \mathcal{B}(\mathcal{H}, \mathcal{H}_i) : i \in \mathbb{I}\}$ is called a (C, C')-controlled K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i : i \in \mathbb{I}\}$ if there exist constants $0 < A \leq B < \infty$, such that

$$A\|K^*f\|^2 \leq \sum_{i\in\mathbb{I}} \langle \Lambda_i Cf, \Lambda_i C'f\rangle \leq B\|f\|^2, \ (f \in \mathcal{H}). \tag{1.1}$$

The numbers A, B are called the lower and upper frame bounds for (C, C')-controlled K-g-frame, respectively. Particularly, if

$$A\|K^*f\|^2 = \sum_{i\in\mathbb{I}} \langle \Lambda_i Cf, \Lambda_i C'f\rangle, \ (f \in \mathcal{H}),$$

then we call $\{\Lambda_i\}_{i\in\mathbb{I}}$ a (C, C')-controlled tight K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i : i \in \mathbb{I}\}$. The (C, C')-controlled tight K-g-frame $\{\Lambda_i\}_{i\in\mathbb{I}}$ is said to be Parseval if $A = 1$. If the right-hand side of (1.1) holds, then $\{\Lambda_i\}_{i\in\mathbb{I}}$ is called a (C, C')-controlled g-Bessel sequence for $\mathcal{H}$ with respect to $\{\mathcal{H}_i : i \in \mathbb{I}\}$.

Suppose that $\{\Lambda_i\}_{i\in\mathbb{I}}$ is a (C, C')-controlled g-Bessel sequence for $\mathcal{H}$ with respect to $\{\mathcal{H}_i : i \in \mathbb{I}\}$. The bounded linear operator $T_{C\Lambda C'} : \left(\bigoplus_{i\in\mathbb{I}} \mathcal{H}_i\right)_{\ell^2} \to \mathcal{H}$ defined as

$$T_{C\Lambda C'}\left(\{f_i\}_{i\in\mathbb{I}}\right) = \sum_{i\in\mathbb{I}} (CC')^{\frac{1}{2}} \Lambda_i^*(f_i), \ (\{f_i\}_{i\in\mathbb{I}} \in (\bigoplus_{i\in\mathbb{I}} \mathcal{H}_i)_{\ell^2}), \tag{1.2}$$

is called the *synthesis operator*. The adjoint operator $T^*_{C\Lambda C'} : \mathcal{H} \to \left(\bigoplus_{i\in\mathbb{I}} \mathcal{H}_i\right)_{\ell^2}$ that is obtained as

$$T^*_{C\Lambda C'}(f) = \left\{\Lambda_i (CC')^{\frac{1}{2}} f\right\}_{i\in\mathbb{I}}, \ (f \in \mathcal{H}), \tag{1.3}$$

is called the *analysis operator*. Composing $T_{C\Lambda C'}$ and $T^*_{C\Lambda C'}$, the operator $S_{C\Lambda C'} : \mathcal{H} \to \mathcal{H}$ given by

$$S_{C\Lambda C'}f = T_{C\Lambda C'}T^*_{C\Lambda C'}f = \sum_{i\in\mathbb{I}} (CC')^{\frac{1}{2}} \Lambda_i^* \Lambda_i (CC')^{\frac{1}{2}} f, \ (f \in \mathcal{H}), \tag{1.4}$$

is called the (C, C')-*controlled g-Bessel sequence operator*. If $\{\Lambda_i\}_{i\in\mathbb{I}}$ is a (C, C')-controlled K-g-frame, then $S_{C\Lambda C'}$ is called the (C, C')-*controlled K-g-frame operator*.

2. Controlled dual K-g-Bessel sequences

As it was mentioned in [4, Lemma 3.3], the frame operator of a controlled g-frame is invertible, but it is not the case for a controlled K-g-frame. Due to this fact, the classical canonical dual for a controlled K-g-frame is absent. This motivates us in this section to introduce the concept of (C, C')-controlled dual K-g-Bessel sequences of a (C, C')-controlled K-g-frame. Moreover, we give some their characterizations by some operator theory tools.

Definition 2.1. Suppose that $K \in \mathcal{B}(\mathcal{H})$, $C, C' \in GL^+(\mathcal{H})$ and $\{\Lambda_i\}_{i\in\mathbb{I}}$ is a (C, C')-controlled K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i : i \in \mathbb{I}\}$. A (C, C')-controlled g-Bessel sequence $\{\Gamma_i\}_{i\in\mathbb{I}}$ for $\mathcal{H}$ with respect to $\{\mathcal{H}_i : i \in \mathbb{I}\}$ is said to be a (C, C')-controlled dual K-g-Bessel sequence of $\{\Lambda_i\}_{i\in\mathbb{I}}$ if

$$Kf = \sum_{i\in\mathbb{I}} (CC')^{\frac{1}{2}} \Lambda_i^* \Gamma_i (CC')^{\frac{1}{2}} f, \ (f \in \mathcal{H}). \tag{2.1}$$

From now on, we consider $C, C' \in GL^+(\mathcal{H})$ and C, C' and S_Λ mutually commute. From [1], we know that the duals of a frame are necessarily frames, but it is not the case for a controlled K-g-frame. The following theorem shows that for any controlled K-g-frame there always exists a controlled dual K-g-Bessel sequence such that they provide a reconstruction formula for any element in the range of K.

Theorem 2.2. *Suppose that $K \in \mathcal{B}(\mathcal{H})$. Then every (C, C')-controlled K-g-frame admits a (C, C')-controlled dual K-g-Bessel sequence.*

The following proposition shows that for any controlled K-g-frame, there is a unique controlled dual K-g-Bessel sequence whose synthesis operator obtains the minimal norm of the set of the norms of synthesis operators of all controlled dual K-g-Bessel sequences of the controlled K-g-frame. The (C, C')-controlled dual K-g-Bessel sequence satisfying in this proposition is called the *canonical (C, C')-controlled dual K-g-Bessel sequence.*

Proposition 2.3. *Suppose that $K \in \mathcal{B}(\mathcal{H})$. Moreover, let $\{\Lambda_i\}_{i\in\mathbb{I}}$ be a (C, C')-controlled K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i : i \in \mathbb{I}\}$.Then, there exists a unique (C, C')-controlled dual K-g-Bessel sequence $\{\Gamma_i\}_{i\in\mathbb{I}}$ of $\{\Lambda_i\}_{i\in\mathbb{I}}$ such that $\|T_{C\Gamma C'}\| \leq \|T_{C\Theta C'}\|$ for any (C, C')-controlled dual K-g-Bessel sequence $\{\Theta_i\}_{i\in\mathbb{I}}$ of $\{\Lambda_i\}_{i\in\mathbb{I}}$.*

3. Canonical controlled Dual K-g-Bessel Sequences for Parseval frames

In this section, we give the exact form of the canonical (C, C')-controlled dual K-g-Bessel sequences for Parseval (C, C')-controlled K-g-frames under the condition that K has closed range.

Proposition 3.1. *Assume that $K \in \mathcal{B}(\mathcal{H})$, $CK = KC$ and $C'K = KC'$. Moreover, let $\{\Lambda_i\}_{i\in\mathbb{I}}$ be a Parseval (C, C')-controlled K-g-frame for $\mathcal{H}$ with* *respect to* $\{\mathcal{H}_i : i \in \mathbb{I}\}$. *Then* $\left\{\Lambda_i (K^{\dagger})^*\right\}_{i\in\mathbb{I}}$ *is a* (C, C')*-controlled dual g-Bessel sequence of* $\{\Lambda_i\}_{i\in\mathbb{I}}$.

Proposition 3.2. *Suppose that* $K \in \mathcal{B}(\mathcal{H})$, $CK = KC$ *and* $C'K = KC'$. *Moreover, let* $\{\Lambda_i\}_{i\in\mathbb{I}}$ *be a Parseval* (C, C')*-controlled K-g-frame for* $\mathcal{H}$ *with respect to* $\{\mathcal{H}_i : i \in \mathbb{I}\}$. *Then the sequence* $\{\Gamma_i \in \mathcal{B}(\mathcal{H}, \mathcal{H}_i) : i \in \mathbb{I}\}$ *is a* (C, C')*-controlled dual g-Bessel sequence of* $\{\Lambda_i\}_{i\in\mathbb{I}}$ *if and only if there exists* $U \in \mathcal{B}(\mathcal{H}, (\bigoplus_{i\in\mathbb{I}} \mathcal{H}_i)_{\ell^2})$ *such that* $T_{C\Lambda C'}U = 0$ *and* $\Gamma_i = \Lambda_i(K^{\dagger})^* + P_iU(CC')^{\frac{-1}{2}}$, *for every* $i \in \mathbb{I}$.

The next result gives the exact form of the canonical (C, C')-controlled dual K-g-Bessel sequence of a Parseval (C, C')-controlled K-g-frame.

Proposition 3.3. *Suppose that* $K \in \mathcal{B}(\mathcal{H})$, $CK = KC$ *and* $C'K = KC'$. *Moreover, let* $\{\Lambda_i\}_{i\in\mathbb{I}}$ *be a Parseval* (C, C')*-controlled K-g-frame for* $\mathcal{H}$ *with respect to* $\{\mathcal{H}_i : i \in \mathbb{I}\}$. *Then* $\left\{\Lambda_i(K^{\dagger})^*\right\}_{i\in\mathbb{I}}$ *is the canonical* (C, C')*-controlled dual K-g-Bessel sequence of* $\{\Lambda_i\}_{i\in\mathbb{I}}$.

Finally, we give a necessary and sufficient condition for a (C, C')-controlled dual K-g-Bessel sequence of a Parseval (C, C')-controlled K-g-frame to be the canonical (C, C')-controlled dual K-g-Bessel sequence.

Proposition 3.4. *Suppose that* $K \in \mathcal{B}(\mathcal{H})$, $CK = KC$ *and* $C'K = KC'$. *Moreover, let* $\{\Lambda_i\}_{i\in\mathbb{I}}$ *be a Parseval* (C, C')*-controlled K-g-frame for* $\mathcal{H}$ *with respect to* $\{\mathcal{H}_i : i \in \mathbb{I}\}$ *and* $\{\Gamma_i\}_{i\in\mathbb{I}}$ *be a* (C, C')*-controlled dual K-g-Bessel sequence of* $\{\Lambda_i\}_{i\in\mathbb{I}}$. *Then* $\{\Gamma_i\}_{i\in\mathbb{I}}$ *is the canonical* (C, C')*-controlled dual K-g-Bessel sequence of* $\{\Lambda_i\}_{i\in\mathbb{I}}$ *if and only if* $T_{C\Gamma C'}T^*_{C\Gamma C'} = T_{C\Gamma C'}T^*_{C\Theta C'}$, *for any* (C, C')*-controlled dual K-g-Bessel sequence* $\{\Theta_i\}_{i\in\mathbb{I}}$ *of* $\{\Lambda_i\}_{i\in\mathbb{I}}$.

References

1. O. Christensen, *An Introduction to Frames and Riesz Bases*, Birkhauser, Boston, 2016.
2. X. Gou, *Canonical dual K-Bessel sequences and dual K-Bessel generators for unitary systems of Hilbert spaces*, J. Math. Anal. Appl. 444 (2016), no. 1, 598–609.
3. D. Hua and Y. Huang, *Controlled K-g-frames in Hilbert spaces*, Results Math., 72 (2017), No. 3, 1227–1238.
4. A. Rahimi and A. Fereydooni, *Controlled G-frames and their G-multipliers in Hilbert spaces*, Analele stiintifice ale Universitatii Ovidius Constanta, Seria Matematica, 21 (2013), No. 2, 223–236. Razmadze Math. Inst., **142** (2006) 134-137.