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EQUIVALENT METRICS ON NORMAL COMPOSITION OPERATORS

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We define some metrics on the set of all bounded normal composition operators in $L^2(\Sigma)$, and show that these metrics are equivalent with the metric induced by the usual operator norm.

1. Introduction and preliminaries

Let $\mathcal{H}$ be a separable, infinite-dimensional, complex Hilbert space and let $B(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. For $A \in B(\mathcal{H})$, let A^*, $\mathcal{R}(A)$, $r(A)$ and $\|A\|$ denote the adjoint, the range, the spectral radius and the usual operator norm of A, respectively. A is called positive if $\langle Ax, x\rangle \geq 0$ holds for each $x \in \mathcal{H}$, in which case we write $A \geq 0$. Let $A \in C(\mathcal{H})$, the subsets of closed and densely defined linear operators on $\mathcal{H}$. Then the defect operator $I + A^*A$ is a bounded and invertible operator on $\mathcal{H}$. The orthogonal projection of $\mathcal{H} \oplus \mathcal{H}$ onto the graph $G(A)$ of $A \in C(\mathcal{H})$ is given by the operator block matrix [21, p. 54]

$$P(A) = \begin{bmatrix} (I + A^*A)^{-1} & A^*(I + AA^*)^{-1} \\ A(I + A^*A)^{-1} & AA^*(I + AA^*)^{-1} \end{bmatrix}.$$

For $A \in C(\mathcal{H})$, put $K(A) = I + A^*A$, $R(A) = (I + A^*A)^{-1}$ and $S(A) = (I + A^*A)^{-1/2}$. The topological structure of $C(\mathcal{H})$ induced by a metric has been considered starting with the paper by Cordes and Labrousse [3]. They proved that the metric distance between two densely defined unbounded operators A and B may be taken as $\|R(A) - R(B)\|$. They showed that this metric defines the same topology for bounded operators as the ordinary metric $\|A - B\|$. Kaufman [12] studied a metric δ on $C(\mathcal{H})$ defined by $\delta(A, B) = \|AS(A) - BS(B)\|$ and discussed connections between δ-convergence and sot-convergence. Also, he showed that this metric is stronger than the gap metric $d(A, B) = \|P(A) - P(B)\|$ (see [11, p. 197]) and not equivalent to it. In [15; 17], Kittaneh and Koliha presented quantitative improvements of the result of Kaufman [12] concerning equivalence of three metrics on the space of bounded linear operators on a Hilbert space. Motivated by the results mentioned above, we define some metrics on the set of all bounded normal composition operators in $L^2(\Sigma)$.

Let (X, Σ, μ) be a complete σ-finite measure space. We use the notation $L^2(\Sigma)$ for $L^2(X, \Sigma, \mu)$. All comparisons between two functions or two sets are to be interpreted as holding up to a μ-null set. We denote the linear space of all complex-valued Σ-measurable functions on X by $L^0(\Sigma)$. The support of a measurable function $u \in L^0(\Sigma)$ is defined by $\sigma(u) = \{x \in X : u(x) \neq 0\}$. Let $\varphi : X \to X$ be a nonsingular measurable point transformation, which means the measure $\mu \circ \varphi^{-1}$, defined by $\mu \circ \varphi^{-1}(B) = \mu(\varphi^{-1}(B))$

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for all $B \in \Sigma$, is absolutely continuous with respect to μ, and write $\mu \circ \varphi^{-1} \ll \mu$. Then by the Radon–Nikodym theorem there exists a unique nonnegative sigma-measurable function h on X with $h = d\mu \circ \varphi^{-1}/d\mu$. Notice that $\sigma(h \circ \varphi) = X$. Let $\varphi^{-1}(\Sigma)$ be a sub-σ-finite algebra of Σ. The conditional expectation operator associated with $\varphi^{-1}(\Sigma)$ is the mapping $f \to E^{\varphi^{-1}(\Sigma)} f$, defined for all μ-measurable nonnegative f where $E^{\varphi^{-1}(\Sigma)} f$ is, by the Radon–Nikodym theorem, the unique finite-valued $\varphi^{-1}(\Sigma)$-measurable function satisfying

$$\int_B f\, d\mu = \int_B E^{\varphi^{-1}(\Sigma)}(f)\, d\mu \quad \text{for all } B \in \varphi^{-1}(\Sigma).$$

For simplicity set $E^{\varphi^{-1}(\Sigma)} = E_\varphi$. As an operator on $L^2(\Sigma)$, E_φ is an orthogonal projection of $L^2(\Sigma)$ onto $L^2(\varphi^{-1}(\Sigma))$. The weighted composition operator W on $\mathcal{D}(W) = \{f \in L^2(\Sigma) : u.(f \circ \varphi) \in L^2(\Sigma)\}$ induced by a measurable function $u \in L^0(\Sigma)$ and a nonsingular self-map measurable function φ is given by $W = M_u C_\varphi$, where M_u is a multiplication operator and C_φ is a composition operator, defined by $M_u f = uf$ and $C_\varphi f = f \circ \varphi$, respectively. Note that the nonsingularity of φ guarantees that C_φ, and so W, is well defined on $\sigma(u)$. It is easy to check that $\|C_\varphi(f)\|_\mu = \|M_{\sqrt{h}} f\|_\mu = \|f\|_{h\,d\mu}$ for all $f \in \mathcal{D}(C_\varphi) = \{f \in L^2(\Sigma) : f \circ \varphi \in L^2(\Sigma)\}$. Hence $\mathcal{D}(C_\varphi) = L^2(\Sigma) \cap L^2(h\, d\mu)$. Moreover, $\overline{\mathcal{D}(C_\varphi)} = L^2(\Sigma)$ if and only if $\mu(\{h = \infty\}) = 0$, and $\overline{\mathcal{R}(C_\varphi)} = L^2(\varphi^{-1}(\Sigma)) = \{f \circ \varphi : f \in L^2(h\, d\mu)\}$. Note that every densely defined composition operator in $L^2(\Sigma)$ is closed; see [2]. A densely defined composition operator C_φ in $L^2(\Sigma)$ is said to be normal if $C_\varphi^* C_\varphi = C_\varphi C_\varphi^*$. A good reference for information on unbounded weighted composition operators is the monograph [1]. Here, we focus on the bounded case. A result of Hoover, Lambert and Quinn [6] shows that $W \in B(L^2(\Sigma))$ if and only if $h E_\varphi(|u|^2) \circ \varphi^{-1} \in L^\infty(\Sigma)$, and in this case, the adjoint W^* of W on $L^2(\Sigma)$ is given by $W^*(f) = h E_\varphi(\bar{u} f) \circ \varphi^{-1}$. Consequently, $C_\varphi \in B(L^2(\Sigma))$ if and only if $h \in L^\infty(\Sigma)$. In this case $\|C_\varphi\|^2 = \|h\|_\infty$ and $L^2(\Sigma) \subseteq L^2(h\, d\mu)$, and so $\mathcal{D}(C_\varphi) = L^2(\Sigma)$. Some other basic facts about bounded composition operators can be found in [5; 22; 23].

Let $A \in B(\mathcal{H})$ with $r(A) > 0$. For $0 < a < r(A)^{-1}$, we shall relate A with a series such as

$$K_a(A) = I + a^2 A^* A + a^4 A^{*^2} A^2 + \cdots$$

and then define $R_a(A)$ and $S_a(A)$. This relation has been previously used by Lambert and Petrovic [19] in the study of spectral reduced algebras; see also [4]. In the next section, we discuss some equivalent metrics on the set $\mathcal{M}$ of all bounded normal composition operators in $L^2(\Sigma)$ endowed with the quasigap metric. More precisely, we define some metrics on $\mathcal{M}$ equivalent to the metric generated by the operator norm. Similar results on densely defined closed operators between Hilbert spaces have been obtained in [3; 10; 18].

2. Equivalent metrics on $\mathcal{M}$

Let $A \in B(\mathcal{H})$ with $r(A) > 0$ and let $0 < a < r(A)^{-1}$ be an arbitrary but fixed number. Define $K_a(A) = \sum_{n=0}^\infty a^{2n} A^{*n} A^n$. Since $\overline{\lim}_{n\to\infty} \|a^{2n} A^{*^n} A^n\|^{1/n} < 1$, the mapping $B(\mathcal{H}) \to B(\mathcal{H})$, $A \mapsto K_a(A)$ is well-defined. Also, for any $x \in \mathcal{H}$ we have

$$\|x\|^2 \le \sum_{n=0}^\infty a^{2n} \|A^n(x)\|^2 = \langle K_a(A)x, x\rangle = \|\sqrt{K_a(A)}x\|^2 \le \|K_a(A)\| \|x\|^2. \tag{2-1}$$

Then $K_a(A)$ is positive and invertible with $\|K_a(A)\| \geq 1$. Set $R_a(A) = K_a^{-1}(A)$ and $S_a(A) = \sqrt{R_a(A)}$. Replacing x by $(K_a(A))^{-1/2}(x)$ in (2-1) we obtain that $\|S_a(A)\| \leq 1$. Thus, $\|R_a(A)\| = \|S_a^2(A)\| \leq 1$. Consequently, $R_a(A)$ and $S_a(A)$ are positive and invertible elements of $B(\mathscr{H})$, $\max\{\|R_a(A)\|, \|S_a(A)\|\} \leq 1$, and

$$\|K_a(A)\| = \sup_{\|x\|=1} \langle K_a(A)x, x\rangle \leq \sum_{n=0}^{\infty} (\|aA\|^2)^n = \frac{1}{1-\|aA\|^2}. \tag{2-2}$$

Let $A_m, A \in B(\mathscr{H})$, $0 < a_0 = \inf\{r(A_m)^{-1}, r(A)^{-1} : m \in \mathbb{N}\}$ and let $0 < a < a_0$. If $\|A_m - A\| \to 0$, then $a^{2n} A_m^{*n} A_m^n \to a^{2n} A^{*n} A^n$ for each $n \in \mathbb{N}$, and so $\|K_a(A_m) - K_a(A)\| \to 0$ as $m \to \infty$. But the converse is not true. Indeed, if A_1 and A_2 are distinct unitary operators on $\mathscr{H}$, then $K_a(A_1) = K_a(A_2) = (1-a^2)^{-1} I$ for all $0 < a < 1$. Set $\mathscr{N} = \{A \in B(\mathscr{H}) \setminus \{0\} : A \text{ is normal}\}$. Let $A \in \mathscr{N}$ and $0 < a < r(A)^{-1} = \|A\|^{-1}$. Then $K_a(A^*) = K_a(A) = K_a(|A|)$, and A^n and A^{*n} commute with $K_a(A)$ and $R_a(A)$. Moreover,

$$K_a(A) = \sum_{n=0}^{\infty} a^{2n} (A^*A)^n = (I - a^2 A^*A)^{-1}, \quad R_a(A) = I - a^2 A^*A, \quad S_a(A) = \sqrt{I - a^2 A^*A}. \tag{2-3}$$

Consequently, $R_a(A) \to 0$, $S_a(A) \to 0$ and $\|K_a(A)\| \to +\infty$ as $a \to \|A\|^{-1}$. Let $A_1, A_2 \in \mathscr{N}$. Then it follows from (2-3) that $K_a(A_1) = K_a(A_2)$ whenever $A_1^*A_1 = A_2^*A_2$, for all $0 < a < \min\{\|A_1\|^{-1}, \|A_2\|^{-1}\}$. Let $0 < a < b < \|A\|^{-1}$. Then $K_a(A) \leq K_b(A)$. Hence the net $\{K_a(A)\}_a$ is increasing with respect to a.

Set $\mathscr{NC} = \{C_\varphi \in B(L^2(\Sigma)) \setminus \{0\} : C_\varphi \text{ is normal}\}$. It is a classical fact that $C_\varphi \in \mathscr{NC}$ if and only if $\varphi^{-1}(\Sigma) = \Sigma$ and $h \circ \varphi = h$; see [5; 22]. In this case, C_φ is injective and has dense range. Moreover, $\|C_\varphi\|^2 = r^2(C_\varphi) = \|h\|_\infty$ and $C_\varphi^* C_\varphi = M_h$. It follows that $K_a(C_\varphi) = M_{(1-a^2h)^{-1}}$, $R_a(C_\varphi) = M_{1-a^2h}$ and $S_a(C_\varphi) = M_{\sqrt{1-a^2h}}$ for all $0 < a < \|C_\varphi\|^{-1}$. Let $A, B, C, D \in B(\mathscr{H})$. Then

$$AB - CD = \tfrac{1}{2}(A-C)(B+D) + \tfrac{1}{2}(A+C)(B-D). \tag{2-4}$$

In the following lemma we recall some useful operator inequalities which will be used later.

Lemma 2.1 (Kittaneh [14; 13; 16]). *Let $A, B \in B(\mathscr{H})$. Then the following hold.*

(a) *If A and B are positive, then $\|A - B\|^2 \leq \|A^2 - B^2\|$.*

(b) *If A and B are positive and $A + B \geq cI > 0$, then $c\|A - B\| \leq \|A^2 - B^2\|$.*

(c) $\|A^*A - B^*B\| \leq \|A - B\|\|A + B\|$.

Theorem 2.2. *Let $\{C_{\varphi_1}, C_{\varphi_2}\} \subseteq \mathscr{NC}$, $0 < a < \min\{\|C_{\varphi_i}\|^{-1} : i = 1, 2\}$ and let $\alpha_a(C_{\varphi_i}) = aC_{\varphi_i} S_a^{-1}(C_{\varphi_i})$. Then $\|\alpha_a(C_{\varphi_1}) - \alpha_a(C_{\varphi_2})\| \leq k_1 \|C_{\varphi_1} - C_{\varphi_2}\|$ for some $k_1 = k_1(a) > 0$.*

Proof. For $i = 1, 2$, put $h_i = d\mu \circ \varphi_i^{-1}/d\mu$. First observe that

$$S_a^{-1}(C_{\varphi_1}) + S_a^{-1}(C_{\varphi_2}) = M_{1/\sqrt{1-a^2h_1} + 1/\sqrt{1-a^2h_2}} \geq 2I.$$

So by Lemma 2.1(b), $2\|S_a^{-1}(C_{\varphi_1})-S_a^{-1}(C_{\varphi_2})\| \le \|S_a^{-2}(C_{\varphi_1})-S_a^{-2}(C_{\varphi_2})\|$. Put $k=\|S_a^{-1}(C_{\varphi_1})+S_a^{-1}(C_{\varphi_2})\|$ and $l=\|C_{\varphi_1}+C_{\varphi_2}\|$. Using (2-2) and (2-4), we obtain

$$\begin{aligned}
\|\alpha_a(C_{\varphi_1})-\alpha_a(C_{\varphi_2})\| &= \|aC_{\varphi_1}S_a^{-1}(C_{\varphi_1})-aC_{\varphi_2}S_a^{-1}(C_{\varphi_2})\| \\
&\le \frac{a}{2}\|(C_{\varphi_1}-C_{\varphi_2})(S_a^{-1}(C_{\varphi_1})+S_a^{-1}(C_{\varphi_2}))\|+\frac{a}{2}\|(C_{\varphi_1}+C_{\varphi_2})(S_a^{-1}(C_{\varphi_1})-S_a^{-1}(C_{\varphi_2}))\| \\
&\le \frac{ak}{2}\|C_{\varphi_1}-C_{\varphi_2}\|+\frac{al}{4}\|S_a^{-2}(C_{\varphi_1})-S_a^{-2}(C_{\varphi_2})\| \\
&= \frac{ak}{2}\|C_{\varphi_1}-C_{\varphi_2}\|+\frac{al}{4}\|S_a^{-2}(C_{\varphi_2})(S_a^{2}(C_{\varphi_1})-S_a^{2}(C_{\varphi_2}))S_a^{-2}(C_{\varphi_1})\| \\
&\le \frac{ak}{2}\|C_{\varphi_1}-C_{\varphi_2}\|+\frac{al}{4}\|S_a^{-2}(C_{\varphi_2})\|\ \|S_a^{-2}(C_{\varphi_1})\|\ \|M_{a(h_1-h_2)}\| \\
&\le \frac{ak}{2}\|C_{\varphi_1}-C_{\varphi_2}\|+\frac{a^2l^2}{4}\|S_a^{-2}(C_{\varphi_1})\|\ \|S_a^{-2}(C_{\varphi_2})\|\ \|C_{\varphi_1}-C_{\varphi_2}\| \\
&= \|C_{\varphi_1}-C_{\varphi_2}\|\left\{\frac{ak}{2}+\frac{a^2l^2}{4}\|K_a(C_{\varphi_1})\|\ \|K_a(C_{\varphi_2})\|\right\} \\
&\le \|C_{\varphi_1}-C_{\varphi_2}\|\left\{\frac{ak}{2}+\frac{a^2l^2}{4(1-a^2\|h_1\|_\infty)(1-a^2\|h_2\|_\infty)}\right\}.
\end{aligned}$$

This completes the proof with

$$k_1=\left\{\frac{ak}{2}+\frac{a^2l^2}{4(1-a^2\|h_1\|_\infty)(1-a^2\|h_2\|_\infty)}\right\}. \qquad \square$$

Notice that $\|h_1+h_2\|_\infty \le \|C_{\varphi_1}+C_{\varphi_2}\|^2 \le 2\|h_1+h_2\|_\infty$ for all $\{C_{\varphi_1}, C_{\varphi_2}\} \subseteq \mathcal{NC}$; see [9].

Lemma 2.3. *Let* $\{C_{\varphi_1}, C_{\varphi_2}\} \subseteq \mathcal{NC}$ *and let* $0<a<\min\{\|C_{\varphi_i}\|^{-1} : i=1,2\}$. *Then*

$$\|S_a(C_{\varphi_1})-S_a(C_{\varphi_2})\| \le k_2\|R_a(C_{\varphi_1})-R_a(C_{\varphi_2})\|$$

for some $k_2>0$.

Proof. Put $a_{\varphi_i}=\sqrt{1-a^2h_i}$. Since $0<a^2h_i \le a^2\|h_i\|_\infty<1$, we get that $\inf_{x\in X} a_{\varphi_i}(x)>0$. Thus, $\min\{a_{\varphi_1}, a_{\varphi_2}\} \ge 1/n_0$ for some $n_0\in\mathbb{N}$. This implies that $a_{\varphi_1}+a_{\varphi_2} \ge 2\min\{a_{\varphi_1}, a_{\varphi_2}\} \ge 2/n_0 := k_2^{-1}$. Then we obtain

$$\begin{aligned}
\|S_a(C_{\varphi_1})-S_a(C_{\varphi_2})\| &= \|M_{a_{\varphi_1}}-M_{a_{\varphi_2}}\| = \|a_{\varphi_1}-a_{\varphi_2}\|_\infty \\
&= \left\|\frac{a_{\varphi_1}^2-a_{\varphi_2}^2}{a_{\varphi_1}+a_{\varphi_2}}\right\|_\infty \le k_2\|M_{a_{\varphi_1}^2}-M_{a_{\varphi_2}^2}\| \\
&\le k_2\|R_a(C_{\varphi_1})-R_a(C_{\varphi_2})\|. \qquad \square
\end{aligned}$$

Theorem 2.4. *Let* $C_{\varphi_i} \in \mathcal{NC}$ *and* $0<a<\min\{\|C_{\varphi_i}\|^{-1} : i=1,2\}$. *Then*

$$\|C_{\varphi_1}-C_{\varphi_2}\| \le k_3\|\alpha_a(C_{\varphi_1})-\alpha_a(C_{\varphi_2})\|$$

for some $k_3=k_3(a)>0$.

Proof. Put $\beta = \|S_a(C_{\varphi_1}) + S_a(C_{\varphi_2}))\|$ and $\gamma = \|\alpha_a(C_{\varphi_1}) + \alpha_a(C_{\varphi_2})\|$. Using equality $K_a(C_{\varphi_i}) - I = M_{a^2h_i/(1-a^2h_i)} = a^2 C^*_{\varphi_i} C_{\varphi_i} S_a^{-2}(C_{\varphi_i})$ and Lemma 2.1(c) we obtain

$$
\begin{aligned}
\text{(2-5)} \qquad \|R_a^{-1}(C_{\varphi_1}) - R_a^{-1}(C_{\varphi_2})\| &= \|(K_a(C_{\varphi_1}) - I) - (K_a(C_{\varphi_2}) - I)\| \\
&= \|a^2 C^*_{\varphi_1} C_{\varphi_1} S_a^{-2}(C_{\varphi_1}) - a^2 C^*_{\varphi_2} C_{\varphi_2} S_a^{-2}(C_{\varphi_2})\| \\
&= \|\alpha^*(C_{\varphi_1})\alpha(C_{\varphi_1}) - \alpha^*(C_{\varphi_2})\alpha(C_{\varphi_2})\| \\
&\le \gamma \|\alpha_a(C_{\varphi_1}) - \alpha_a(C_{\varphi_2})\|.
\end{aligned}
$$

Then by Lemma 2.3 and (2-5) we have

$$
\begin{aligned}
\|C_{\varphi_1} - C_{\varphi_2}\| &= \frac{1}{a}\|(aC_{\varphi_1} S_a^{-1}(C_{\varphi_1}))S_a(C_{\varphi_1}) - (aC_{\varphi_2} S_a^{-1}(C_{\varphi_2}))S_a(C_{\varphi_2})\| \\
&= \frac{1}{a}\|\alpha_a(C_{\varphi_1})S_a(C_{\varphi_1}) - \alpha_a(C_{\varphi_2})S_a(C_{\varphi_2})\| \\
&\le \frac{\beta}{2a}\|\alpha_a(C_{\varphi_1}) - \alpha_a(C_{\varphi_2})\| + \frac{\gamma}{2a}\|S_a(C_{\varphi_1}) - S_a(C_{\varphi_2}))\| \\
&\le \frac{\beta}{2a}\|\alpha_a(C_{\varphi_1}) - \alpha_a(C_{\varphi_2})\| + \frac{\gamma k_2}{2a}\|R_a(C_{\varphi_1}) - R_a(C_{\varphi_2})\| \\
&\le \frac{\beta}{2a}\|\alpha_a(C_{\varphi_1}) - \alpha_a(C_{\varphi_2})\| + \frac{\gamma k_2}{2a}\|R_a(C_{\varphi_1})(R_a^{-1}(C_{\varphi_1}) - R_a^{-1}(C_{\varphi_2}))R_a(C_{\varphi_2})\| \\
&\le \frac{\beta}{2a}\|\alpha_a(C_{\varphi_1}) - \alpha_a(C_{\varphi_2})\| + \frac{\gamma^2 k_2}{2a}\|R_a(C_{\varphi_1})\|\|R_a(C_{\varphi_2})\|\|\alpha_a(C_{\varphi_1}) - \alpha_a(C_{\varphi_2})\| \\
&= \|\alpha_a(C_{\varphi_1}) - \alpha_a(C_{\varphi_2})\|\left\{\frac{\beta}{2a} + \frac{\gamma^2 k_2}{2a}\|R_a(C_{\varphi_1})\|\|R_a(C_{\varphi_2})\|\right\}.
\end{aligned}
$$

Since $\|R_a(C_{\varphi_i})\| \le 1$, then the desired conclusion holds with $k_3 = \{\beta/2a + \gamma^2 k_2/2a\}$. □

Lemma 2.5. *Let $C_{\varphi_i} \in \mathcal{NC}$, $0 < a < \min\{\|C_{\varphi_i}\|^{-1} : i = 1, 2\}$ and $0 < u \in L^\infty(\Sigma)$. Then $C_{\varphi_1} = C_{\varphi_2}$ whenever $M_u C_{\varphi_1} = C_{\varphi_2}$.*

Proof. It suffices to show that $u = 1$. If $\mu(X) < \infty$, then there is nothing to prove, because $C_{\varphi_1}(1) = C_{\varphi_2}(1) = 1$. Set $A = \{x \in \sigma(u) : u(x) \ne 1\}$. If $\mu(A) > 0$, then there exists $B \subseteq A$ with $0 < \mu(B) < \infty$. Moreover, since ${\varphi_1}^{-1}(\Sigma) = \Sigma$, then $B = {\varphi_1}^{-1}(C)$ for some $C \in \Sigma$. Now choose $C_0 \subseteq C$ such that $\mu(C_0) < \infty$ and $\mu({\varphi_1}^{-1}(C_0)) > 0$. Take $f_0 = \chi_{C_0}$. Then $u\chi_{{\varphi_1}^{-1}(C_0)} = M_u C_{\varphi_1}(f_0) = C_{\varphi_2}(f_0) = \chi_{{\varphi_2}^{-1}(C_0)}$. But this is a contraction. Thus, $\mu(A) = 0$ and hence $C_{\varphi_1} = C_{\varphi_2}$. □

Now we consider the bounded weighted composition operators on $L^2(\Sigma)$. Recall that the adjoint W^* of W is given by $W^*(f) = hE(\bar{u} f) \circ \varphi^{-1}$ for each $f \in L^2(\Sigma)$. As an application of this adjoint formula, we have $W^*W = M_J$, where $J = hE(|u|^2) \circ \varphi^{-1}$. Moreover, W is normal (see [2]) if and only if $\varphi^{-1}(\Sigma) \cap J = \Sigma$ and $J = J \circ \varphi$ on $\sigma(J)$. Put $\mathcal{NW} = \{M_u C_\varphi \in B(L^2(\Sigma)) \setminus \{0\} : M_u C_\varphi \text{ is normal}\}$. Suppose $\{W_n\} \subseteq \mathcal{NW}$ converges (in norm) to some $K \in B(L^2(\Sigma))$. Then $\{W_n^*\}$ converges to K^*, and since the multiplication map is continuous, then we have $K^*K = \lim_{n\to\infty} W_n^* W_n = \lim_{n\to\infty} W_n W_n^* = KK^*$, and so K is normal. Let $W = M_u C_\varphi \in \mathcal{NW}$ and let $0 < a < \|W\|^{-1}$ be a fixed number. Direct computations show that $K_a(W) = M_{(1-a^2J)^{-1}}$, $R_a(W) = M_{(1-a^2J)}$ and $S_a(W) = M_{\sqrt{1-a^2J}}$. The previous results can be stated in terms of weighted composition operators.

Definition 2.6. Let $\mathcal{M} \subseteq \mathcal{NW}$. We say that $\mathcal{M}$ has infimum property if

$$a_0 := \inf\{\|M_u C_\varphi\|^{-1} : M_u C_\varphi \in \mathcal{M}\} > 0.$$

Let $C_{\varphi_i} \in \mathcal{M}$. For fixed $0 < a < a_0$, let $a < a_1 < a_0$. It is easy to see that $(1 - a_1^2\|h_i\|) \leq \|a_{\varphi_i}\|_\infty^2 \leq 2$, where $h_i = h_{\varphi_i}$ and $a_{\varphi_i} = \sqrt{1 - a^2 h_i}$. Define

$$\delta_a(C_{\varphi_1}, C_{\varphi_2}) = \|\alpha_a(C_{\varphi_1}) - \alpha_a(C_{\varphi_2})\|, \quad \text{where } \alpha_a(C_{\varphi_i}) = aC_{\varphi_i} S_a^{-1}(C_{\varphi_i}) = M_{a/\sqrt{1-a^2 h_i}} C_{\varphi_i}.$$

Note that $\alpha_a(C_{\varphi_i})$ is not necessarily a contraction. Indeed,

$$\|\alpha_a(C_{\varphi_i})\|^2 = a^2 \left\| \frac{a h_i}{1 - a^2 h_i} \right\|_\infty.$$

Moreover, $\delta_a(C_{\varphi_1}, C_{\varphi_2}) = 0$ implies that $M_u C_{\varphi_1} = C_{\varphi_2}$, where $u = a_{\varphi_2}/a_{\varphi_1}$. Then by Lemma 2.5, $C_{\varphi_1} = C_{\varphi_2}$. Thus, for each correspondence a, the function δ_a is a metric on $\mathcal{M}$. Put

$$l = \|C_{\varphi_1} + C_{\varphi_2}\|, \quad \gamma = \|\alpha_a(C_{\varphi_1}) + \alpha_a(C_{\varphi_2})\|, \quad p = \|K_a(C_{\varphi_1}) + K_a(C_{\varphi_2}))\|.$$

Then by Theorem 2.2, Theorem 2.4 and (2-5) we have

$$\delta_a(C_{\varphi_1}, C_{\varphi_2}) \leq k_1\|C_{\varphi_1} - C_{\varphi_2}\| \leq k_1 k_3 \delta_a(C_{\varphi_1}, C_{\varphi_2}), \tag{2-6}$$

$$\|K_a(C_{\varphi_1}) - K_a(C_{\varphi_2})\| \leq \gamma \delta_a(C_{\varphi_1}, C_{\varphi_2}) \leq \gamma k_1 \|C_{\varphi_1} - C_{\varphi_2}\|. \tag{2-7}$$

Moreover, since $K_a(C_{\varphi_i})$ and $\alpha_a(C_{\varphi_i})$ are bounded and positive, then by (2-5), (2-6) and (2-7) we get

$$\begin{aligned} \|C_{\varphi_1}^* K_a(C_{\varphi_1}) - C_{\varphi_2}^* K_a(C_{\varphi_2})\| &= \|C_{\varphi_1} K_a(C_{\varphi_1}) - C_{\varphi_2} K_a(C_{\varphi_2})\| \\ &\leq \frac{p}{2}\|C_{\varphi_1} - C_{\varphi_2}\| + \frac{l}{2}\|K_a(C_{\varphi_1}) - K_a(C_{\varphi_2})\| \\ &\leq \frac{p}{2}\|C_{\varphi_1} - C_{\varphi_2}\| + \frac{l\gamma k_1}{2}\|C_{\varphi_1} - C_{\varphi_2}\|, \end{aligned} \tag{2-8}$$

$$\begin{aligned} \|a^2 C_{\varphi_1}^* C_{\varphi_1} K_a(C_{\varphi_1}) - a^2 C_{\varphi_2}^* C_{\varphi_2} K_a(C_{\varphi_2})\| &= \|\alpha^*(C_{\varphi_1})\alpha(C_{\varphi_1}) - \alpha^*(C_{\varphi_2})\alpha(C_{\varphi_2})\| \\ &\leq \gamma \|\alpha_a(C_{\varphi_1}) - \alpha_a(C_{\varphi_2})\| \\ &\leq \gamma k_1 \|C_{\varphi_1} - C_{\varphi_2}\|. \end{aligned} \tag{2-9}$$

So, we have the following corollary.

Corollary 2.7. *In $\mathcal{M}$, the metric δ_a is equivalent to the metric generated by the operator norm.*

Let $C_\varphi \in \mathcal{M}$. Since for each $n \in \mathbb{N}$, $|C_\varphi|^{2n} = (C_\varphi^*)^n C_\varphi^n$, we have $K_a(|C_\varphi|) = K_a(C_\varphi)$ and hence $S_a^{-1}(|C_\varphi|) = S_a^{-1}(C_\varphi)$. Consequently, $|\alpha_a(C_\varphi)| = \alpha_a(|C_\varphi|)$.

Now, let $\{C_{\varphi_n}\} \subseteq \mathcal{M}$ and $\delta_a(C_{\varphi_n}, C_\varphi) \to 0$ as $n \to \infty$. Then by (2-6) we have $\|C_{\varphi_n} - C_\varphi\| \to 0$. But

$$\||C_{\varphi_n}| - |C_\varphi|\| = \|M_{\sqrt{h_{\varphi_n}} - \sqrt{h_\varphi}}\| \le \left\| \frac{1}{\sqrt{h_{\varphi_n}} + \sqrt{h_\varphi}} \right\|_\infty \|M_{h_{\varphi_n} - h_\varphi}\|$$
$$= \left\| \frac{1}{\sqrt{h_{\varphi_n}} + \sqrt{h_\varphi}} \right\|_\infty \|C^*_{\varphi_n} C_{\varphi_n} - C^*_\varphi C_\varphi\|$$
$$\le \left\| \frac{1}{\sqrt{h_{\varphi_n}} + \sqrt{h_\varphi}} \right\|_\infty \|C_{\varphi_n} - C_\varphi\| \, \|C_{\varphi_n} + C_\varphi\|.$$

Again by using Corollary 2.7, we conclude that $\delta_a(|C_{\varphi_n}|, |C_\varphi|) \to 0$. It is not the case in general that $\|A_n - A\| \to 0$ whenever $\||A_n| - |A|\| \to 0$. Indeed, for

$$A_n = \begin{bmatrix} (-1)^n & 0 \\ 0 & 1 \end{bmatrix}$$

and $A = I$, we have $\||A_n| - |A|\| = 0$ but $\|A_{2n+1} - A\| = 2$. However, in our setting, if

$$\max\{\|\sqrt{h_n} - \sqrt{h}\|_\infty, \|M_{1/\sqrt{h_n}} C_{\varphi_n} - M_{1/\sqrt{h}} C_\varphi\|\} \to 0 \quad \text{as } n \to \infty,$$

then we have

$$\|C_{\varphi_n} - C_\varphi\| = \|M_{\sqrt{h_n}}(M_{1/\sqrt{h_n}} C_{\varphi_n}) - M_{\sqrt{h}}(M_{1/\sqrt{h}} C_\varphi)\|$$
$$\le \|\sqrt{h_n} - \sqrt{h}\|_\infty \|M_{1/\sqrt{h_n}} C_{\varphi_n} + M_{1/\sqrt{h}} C_\varphi\| + \|M_{1/\sqrt{h_n}} C_{\varphi_n} - M_{1/\sqrt{h}} C_\varphi\| \|\sqrt{h_n} + \sqrt{h}\|_\infty \to 0.$$

So we have the following corollary.

Corollary 2.8. *Let $\{C_{\varphi_n}, C_\varphi\} \subseteq \mathcal{M}$. Then the following hold.*

(a) $\delta_a(|C_{\varphi_n}|, |C_\varphi|) \to 0$ *whenever* $\delta_a(C_{\varphi_n}, C_\varphi) \to 0$; *i.e., the mapping $C_\varphi \mapsto |C_\varphi|$ is continuous on* $(\mathcal{M}, \delta_a)$.

(b) *If* $\max\{\||C_{\varphi_n}| - |C_\varphi|\|, \|M_{1/\sqrt{h_n}} C_{\varphi_n} - M_{1/\sqrt{h}} C_\varphi\|\} \to 0$, *then* $\delta_a(|C_{\varphi_n}|, |C_\varphi|) \to 0$.

For $C_\varphi \in B(L^2(\Sigma))$ with $0 < a < \|C_\varphi\|^{-1}$, the graph of C_φ is the set $\mathcal{G}(C_\varphi) = \{(f, C_\varphi(f)) : f \in L^2(\Sigma)\}$. Now, let $C_\varphi \in \mathcal{M}$. Define

$$\mathcal{P}_a(C_\varphi) = \begin{bmatrix} K_a(C_\varphi) & -a^2 C^*_\varphi K_a(C_\varphi) \\ C_\varphi K_a(C_\varphi) & -a^2 C^*_\varphi C_\varphi K_a(C_\varphi) \end{bmatrix}.$$

Then $\mathcal{P}_a(C_\varphi)$ is a bounded operator on $L^2(\Sigma) \oplus L^2(\Sigma)$ with $\mathcal{R}(\mathcal{P}_a(C_\varphi)) \subseteq \mathcal{G}(C_\varphi)$. Let $A, B, C, D \in B(L^2(\Sigma))$ and $k = \|A\|^2 + \|B\|^2 + \|C\|^2 + \|D\|^2$. Put

$$P_n = \begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix}, \qquad P = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

We recall the well-known fact (see e.g. [3]) that

$$\|P_n - P\| \to 0 \quad \Longleftrightarrow \quad \max\{\|A_n - A\|, \|B_n - B\|, \|C_n - C\|, \|D_n - D\|\} \to 0.$$

Moreover, if P is positive, then

$$\max\{\sqrt{\|A\|^2+\|C\|^2}, \sqrt{\|B\|^2+\|D\|^2}\} \le \|P\| \le \sqrt{k} \le 2\|P\|. \tag{2-10}$$

Now, we consider two other metrics on $\mathcal{M}$ defined as

$$d_a(C_{\varphi_1}, C_{\varphi_2}) = \|\mathcal{P}_a(C_{\varphi_1}) - \mathcal{P}_a(C_{\varphi_2})\|,$$
$$l_a(C_{\varphi_1}, C_{\varphi_2}) = \sqrt{2\|K_a(C_{\varphi_1}) - K_a(C_{\varphi_2})\|^2 + b\|C_{\varphi_1}K_a(C_{\varphi_1}) - C_{\varphi_2}K_a(C_{\varphi_2})\|^2},$$

where $b = a^4+1$. The metric d_a is called, in that case, a quasigap metric, or more specifically, an a-quasigap metric. Inspired by the matrix representation of $\mathcal{P}_a(C_{\varphi_i})$ we get that

$$\begin{aligned} d_a^2(C_{\varphi_1}, C_{\varphi_2}) &\le \|K_a(C_{\varphi_1}) - K_a(C_{\varphi_2})\|^2 + a^4\|C^*_{\varphi_1}K_a(C_{\varphi_1}) - C^*_{\varphi_2}K_a(C_{\varphi_2})\|^2 + \|C_{\varphi_1}K_a(C_{\varphi_1}) - C_{\varphi_2}K_a(C_{\varphi_2})\|^2 \\ &\quad + \|\alpha^*(C_{\varphi_1})\alpha(C_{\varphi_1}) - \alpha^*(C_{\varphi_2})\alpha(C_{\varphi_2})\|^2. \end{aligned} \tag{2-11}$$

Using (2-7), (2-8) and (2-9) we obtain

$$d_a(C_{\varphi_1}, C_{\varphi_2}) \le c\|C_{\varphi_1} - C_{\varphi_2}\| \tag{2-12}$$

for some $c > 0$. On the other hand we have

$$\begin{aligned} \delta_a(C_{\varphi_n}, C_\varphi) &= \|\alpha_a(C_{\varphi_n}) - \alpha_a(C_\varphi)\| \\ &\le \|aC_{\varphi_n}K_a(C_{\varphi_n})K_a^{-1/2}(C_{\varphi_n}) - C_\varphi K_a(C_\varphi)K_a^{-1/2}(C_\varphi)\| \\ &\le a\|C_{\varphi_n}K_a(C_{\varphi_n}) - C_\varphi K_a(C_\varphi)\|\|K_a^{-1/2}(C_{\varphi_n}) + K_a^{-1/2}(C_\varphi)\| \\ &\qquad + a\|C_{\varphi_n}K_a(C_{\varphi_n}) + C_\varphi K_a(C_\varphi)\|\|K_a^{-1/2}(C_{\varphi_n}) - K_a^{-1/2}(C_\varphi)\|. \end{aligned}$$

But using Lemma 2.1(a) we have

$$\begin{aligned} \|K_a^{-1/2}(C_{\varphi_n}) - K_a^{-1/2}(C_\varphi)\|^2 &\le \|K_a^{-1}(C_{\varphi_n}) - K_a^{-1}(C_\varphi)\| \\ &= \|K_a^{-1}(C_{\varphi_n})(K_a(C_{\varphi_n}) - K_a(C_\varphi))K_a^{-1}(C_\varphi)\| \\ &\le \|R_a(C_{\varphi_n})\|\|R_a(C_\varphi)\|\|K_a(C_{\varphi_n}) - K_a(C_\varphi)\| \\ &\le \|K_a(C_{\varphi_n}) - K_a(C_\varphi)\| \\ &\le d_a(C_{\varphi_n}, C_\varphi), \end{aligned}$$

and $\|C_{\varphi_n}K_a(C_{\varphi_n}) - C_\varphi K_a(C_\varphi)\| \le d_a(C_{\varphi_n}, C_\varphi)$. Moreover, since $S_a(C_{\varphi_n})$ and $S_a(C_\varphi)$ are contractions,

$$\|K_a^{-1/2}(C_{\varphi_n}) + K_a^{-1/2}(C_\varphi)\| \le 2.$$

So, if $d_a(C_{\varphi_n}, C_\varphi) \to 0$ as $n \to \infty$, then $\delta_a(C_{\varphi_n}, C_\varphi) \to 0$. On the other hand, by (2-5),

$$\|\alpha^*(C_{\varphi_1})\alpha(C_{\varphi_1}) - \alpha^*(C_{\varphi_2})\alpha(C_{\varphi_2})\| = \|K_a(C_{\varphi_1}) - K_a(C_{\varphi_2})\|,$$

so by using (2-11) we get that

$$l_a(C_{\varphi_1}, C_{\varphi_2}) \le \sqrt{2}d_a(C_{\varphi_1}, C_{\varphi_2}) \le \sqrt{2}l_a(C_{\varphi_1}, C_{\varphi_2}).$$

In view of these observations and Corollary 2.7 we have the following corollary.

Corollary 2.9. (i) *In $\mathcal{M}$, the metrics d_a, δ_a, l_a and the metric generated by the operator norm on $\mathcal{M}$ are equivalent.*

(ii) *The mappings $C_\varphi \mapsto K_a(C_\varphi)$, $C_\varphi \mapsto R_a(C_\varphi)$ and $C_\varphi \mapsto S_a(C_\varphi)$ on $\mathcal{M}$ with operator norm are continuous.*

Let $B_C(\mathcal{H})$ be the set of all bounded linear operators on $\mathcal{H}$ with closed range. For $T \in B_C(\mathcal{H})$, the Moore–Penrose inverse of T, denoted by $T^\dagger$, is the unique operator $T^\dagger \in B_C(\mathcal{H})$ that satisfies the equations $TST = T$, $STS = S$, $(TS)^* = TS$ and $(ST)^* = ST$. The reduced minimum modulus $\gamma(T)$ of $T \in B(\mathcal{H})$ is defined by $\gamma(T) = \inf\{\|Tx\| : \operatorname{dist}(x, \mathcal{N}(T)) = 1 \text{ for all } x \in \mathcal{H}\}$. Notice that $\gamma(T) = \|T^\dagger\|^{-1}$.

Lemma 2.10 [22; 8]. *Let $C_\varphi \in B(L^2(\Sigma))$. Then the following hold.*

(i) *$C_\varphi \in B_C(L^2(\Sigma))$ if and only if $h = d\mu \circ \varphi^{-1}/d\mu$ is bounded away from zero on $\sigma(h)$.*

(ii) *$C_\varphi C_\varphi^* = M_{h\circ\varphi} E$, where $E = E^{\varphi^{-1}(\Sigma)}$.*

(iii) *If $C_\varphi \in B_C(L^2(\Sigma))$, then $C_\varphi^\dagger = M_{\chi_{\sigma(h)}/h} C_\varphi^*$.*

(iv) *$\gamma(C_\varphi) = \|1/(h \circ \varphi)\|_\infty^{-1/2}$.*

Using Lemma 2.10 and the fact that $\chi_{\sigma(h)} \circ \varphi = \chi_{\varphi^{-1}(\sigma(h))} = \chi_{\sigma(h\circ\varphi)} = 1$, we obtain

$$C_\varphi C_\varphi^\dagger = C_\varphi M_{\chi_{\sigma(h)}/h} C_\varphi^* = M_{1/(h\circ\varphi)} C_\varphi C_\varphi^* = E.$$

Put $\mathcal{CM} = \mathcal{M} \cap \mathcal{NC}$ and $E_i = E^{\varphi_i^{-1}(\Sigma)}$. Define $d_{ma} : \mathcal{CM} \times \mathcal{CM} \to \mathbb{R}^+$ as

$$d_{ma}(C_{\varphi_1}, C_{\varphi_2}) = \sqrt{\|E_1 - E_2\|^2 + \|\alpha_a(C_{\varphi_1}) - \alpha_a(C_{\varphi_2})\|^2}.$$

Then d_{ma} is a metric on closed subset $\mathcal{CM}$ of $B(L^2(\Sigma))$. Let $\{C_{\varphi_n}, C_\varphi\} \subseteq \mathcal{CM}$. Using Corollary 2.7, $\|C_{\varphi_n} - C_\varphi\| \to 0$ whenever $d_{ma}(C_{\varphi_n}, C_\varphi) \to 0$ as $n \to \infty$. Now, let $\|C_{\varphi_n} - C_\varphi\| \to 0$. Then by Corollary 2.7 and using the continuity of the map $C_\varphi \mapsto C_\varphi^\dagger$ (see [7]) on $B_C(L^2(\Sigma))$ we have $\|\alpha_a(C_{\varphi_n}) - \alpha_a(C_\varphi)\| \to 0$, $\|C_{\varphi_n}^\dagger - C_\varphi^\dagger\| \to 0$, and so $\|E_n - E\| = \|C_{\varphi_n} C_{\varphi_n}^\dagger - C_\varphi C_\varphi^\dagger\| \to 0$. Consequently, $d_{ma}(C_{\varphi_n}, C_\varphi) \to 0$. Moreover, $\|C_{\varphi_n} - C_\varphi\| \to 0$ implies that $\gamma(C_{\varphi_n}) = \|C_{\varphi_n}^\dagger\|^{-1} \to \|C_\varphi^\dagger\|^{-1} = \gamma(C_\varphi)$. Note that $\|C_{\varphi_n} - C_\varphi\| \to 0$ does not imply that $\|E_n - E\| \to 0$. However, if $\max\{\|h\|_\infty^{-1}, \|h_n\|_\infty^{-1}\} \le k^2$ for some positive integer k, then $\|C_{\varphi_n} C_{\varphi_n}^\dagger - C_\varphi C_\varphi^\dagger\| \le k\|C_{\varphi_n} - C_\varphi\|$; see [20, Proposition 6.2]. Thus, in this case, $\|E_n - E\| \to 0$ wherever $\|C_{\varphi_n} - C_\varphi\| \to 0$.

Corollary 2.11. *For a fixed positive integer k, the metric d_{ma} is equivalent to the metric generated by the operator norm on $\{C_{\varphi_i} \in \mathcal{CM} : \|h_i\|^{-1} \le k^2\}$.*

For $0 < a < \|C_\varphi\|^{-1}$, the a-bisecting of $C_\varphi \in \mathcal{M}$ is, in our case, defined as

$$(\widetilde{C}_\varphi)_a = a S_a^{-1}(C_\varphi)(I + S_a^{-1}(C_\varphi))^{-1} C_\varphi.$$

Put $a_\varphi = \sqrt{1 - a^2 h}$. Then $(\widetilde{C}_\varphi)_a = M_{a(1+a_\varphi)^{-1}} C_\varphi$ is a normal weighted composition operator with norm $\|(\widetilde{C}_\varphi)_a\| = \|(1 + a_\varphi)^{-1}\sqrt{a^2 h}\|_\infty$. Let $C_{\varphi_i} \in \mathcal{M}$ and $0 < a < a_0$; see Definition 2.6. If $(\widetilde{C}_{\varphi_1})_a = (\widetilde{C}_{\varphi_2})_a$, then $M_u C_{\varphi_1} = C_{\varphi_2}$ where $u = (1 + a_{\varphi_1})/(1 + a_{\varphi_2})$. Then by Lemma 2.5, $C_{\varphi_1} = C_{\varphi_2}$. Thus, the mapping $C_\varphi \mapsto (\widetilde{C}_\varphi)_a$ from $\mathcal{M}$ into $\mathcal{NW}$ is injective and continuous with respect to the operator norm. Indeed, if $\|C_{\varphi_n} - C_\varphi\| \to 0$, then $M_{h_n} = C_{\varphi_n}^* C_{\varphi_n} \to C_\varphi^* C_\varphi = M_h$, and so $\|h_n - h\|_\infty \to 0$. Then $a_{\varphi_n} \to a_\varphi$ and hence $M_{a(1+a_{\varphi_n})^{-1}} \to M_{a(1+a_\varphi)^{-1}}$. Now, the desired conclusion follows from the continuity of the multiplication map in $B(L^2(\Sigma))$. The bisecting of a closed operator A was originally introduced in [18] by Labrousse and Mercier, in order to study semi-Fredholm operators.

Let $C_{\varphi_1}, C_{\varphi_2} \in \mathcal{M}$. For a fixed $0 < a < a_0$ we define the Cordes–Labrousse type transform $V_{1,2}$ with respect to the pair $(C_{\varphi_1}, C_{\varphi_2})$ as

$$V_{1,2} = M_{1/(a_{\varphi_1} a_{\varphi_2})} - (aM_{1/a_{\varphi_1}} C^*_{\varphi_1})(aM_{1/a_{\varphi_2}} C_{\varphi_2}).$$

Then $V_{1,2} \in B(L^2(\Sigma))$, $V^*_{1,1} = I$ and $V^*_{1,2} = V_{2,1}$. For $f \in L^2(\Sigma)$, set $f_1 = M_{1/a_{\varphi_2}} f$ and $f_2 = aM_{1/a_{\varphi_2}} C_{\varphi_2} f$. By a similar argument to that used in [18, Lemma 5.3], we can show that

$$|\|V_{1,2} f\|^2 - \|f\|^2| \le (\|f_1\|^2 + \|f_2\|^2) l_a(C_{\varphi_1}, C_{\varphi_2})$$

and

$$\|f_1\|^2 + \|f_2\|^2 = \int_X \frac{1 + a^2 h_2}{1 - a^2 h_1} |f|^2 \, d\mu \le 2\|K_a(C_{\varphi_2})\| \|f\|^2.$$

It follows that $(1 - 2\|K_a(C_{\varphi_2})\| \; l_a(C_{\varphi_1}, C_{\varphi_2}))\|f\|^2 \le \|V_{1,2} f\|^2$. In view of these observations we have the following proposition.

Proposition 2.12. *For a fixed $0 < a < a_0$, the following assertions hold.*

(i) *The bisecting map $C_\varphi \mapsto (\widetilde{C}_\varphi)_a$ from $\mathcal{M}$ into $\mathcal{NW}$ is injective and continuous.*

(ii) *If $l_a(C_{\varphi_1}, C_{\varphi_2})) < 1/(2\|K_a(C_{\varphi_2})\|)$, then the Cordes–Labrousse type transform $V_{1,2}$ is invertible.*

Example 2.13. Let $X = [0, 1]$ with Lebesgue measure, $a_n, k \in (1, +\infty)$ and let $\varphi_n(x) = a_n x + b$ and $\varphi(x) = kx + b$. Then, for all $n \in \mathbb{N}$, C_{φ_n} is a bounded and normal composition operator on $L^2([0, 1])$ with $\|C_{\varphi_n}\| = 1/\sqrt{a_n}$ and $C^*_{\varphi_n} = (1/a_n) C_{\varphi_n^{-1}}$. Moreover, $1 \le a_0 = \inf\{\sqrt{a_n} : n \ge 1\}$ and for each $0 < a < 1$, the operator $K_a(C_{\varphi_n}) = (a_n/(a_n - a^2))I$ is

scalar multiple of the identity operator. Thus, for fixed $n_0 \in \mathbb{N}$, $\|K_a(C_{\varphi_{n_0}})\| \to +\infty$ as $a \to \sqrt{a_{n_0}}$. Put $b_n = a_n/(a_n - a^2)$. Then

$$\mathcal{P}_a(C_{\varphi_n}) = \begin{bmatrix} b_n I & (-a^2 b_n/a_n) C_{\varphi_n^{-1}} \\ b_n C_{\varphi_n} & (-a^2 b_n/a_n) I \end{bmatrix}.$$

If $\|C_{\varphi_n} - C_\varphi\| \to 0$, then $h_n = 1/a_n = \|C_{\varphi_n}\|^2 \to \|C_\varphi\|^2 = 1/k = h$. Conversely, let $h_n \to h$. It follows that $\varphi_n \to \varphi$. Then, by the Weierstrass approximation theorem, $f \circ \varphi_n \to f \circ \varphi$ uniformly on $[0, 1]$ for all continuous functions $f \in C([0, 1])$. Since $C([0, 1])$ is dense in $L^2([0, 1])$ and $\|C_{\varphi_n}\| \le 1/a_0$ for all $n \in \mathbb{N}$, we have $C_{\varphi_n}(f) \to C_\varphi(f)$ in L^2-norm for all $f \in L^2([0, 1])$. Also, according to the previous discussions we have

$$(\widetilde{C}_\varphi)_a = M_{a/\left(\sqrt{a_n}(1+\sqrt{(a_n - a^2)/a_n})\right)} C_\varphi, \quad \|(\widetilde{C}_\varphi)_a\| = \frac{\|aC_\varphi\|}{1 + \sqrt{1 - \|aC_\varphi\|^2}},$$

$$(V_{1,2} f)(x) = \frac{1}{c} f(x) - \frac{a^2}{a_1 c} f\left(\frac{a_2}{a_1} x + b\left(\frac{a_1 - a_2}{a_1}\right)\right),$$

where

$$c = \sqrt{\left(\frac{a_1 - a^2}{a_1}\right)\left(\frac{a_2 - a^2}{a_2}\right)}.$$

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