The 7th Seminar on Functional Analysis and its Applications
1-2 March 2023
Imam Khomeini International University
Qazvin, Iran

ON ELEMENTS OF SECOND DUAL OF A HYPERGROUP ALGEBRA

ALI GHAFFARI

Department of Mathematics, Semnan University, P.O.Box 35195-363, Semnan, Iran
aghaffari@semnan.ac.ir

Abstract. Let H be a locally compact hypergroup with left invariant Haar measure and let $L^p(H)$, $1 \leq p < \infty$, be the complex Lebesgue space associated with it. Let $L^\infty(H)$ be the set of all locally measurable functions that are bounded except on a locally null set, modulo functions that are zero locally a.e. Let $\mu \in M(H)$. We want to find out when $\mu F \in L^1(H)$ implies that $F \in L^1(H)$. Some necessary and sufficient conditions is found for a measure μ for which if $\mu F \in L^1(H)$ for every $F \in L^\infty(H)^*$, then $F \in L^1(H)$.

1. Introduction

Hypergroups are locally compact spaces whose bounded Radon measures form an algebra which has similar properties to the convolution measures algebra of a locally compact group. Locally compact hypergroups were independently introduced around the 1970's by Dunkl, Jewett and Spector. They generalize the concepts of locally compact groups with the purpose of doing standard harmonic analysis. For the theory of hypergroups and most of the basic properties we refer to [2], [4] and [5].

Let H be a locally compact Hausdorff space. Let $M(H)$ be the space of complex-valued, regular Borel measures on H. We denote by $M^1(H)$ the convex set formed by the probability measures on H. The support of a measure μ is denoted by suppμ. Let $\mathcal{C}(H)$ be the space of all compact

2020 *Mathematics Subject Classification. Primary 43A15; Secondary 43A22*

Key words and phrases. Banach algebras, Discerete topology, Hypergroup algebras, Second dual of hypergroup algebras.

subsets of H. The triple $(M(H), +, *)$ will be called a hypergroup if the following conditions are satisfied.

(1) the vector space $(M(H), +)$ admits a binary operation $*$ under which it is an algebra,
(2) for $x, y \in H$, $\delta_x * \delta_y$ is a probability measure on H with compact support,
(3) the mapping $(x, y) \mapsto \delta_x * \delta_y$ of $H \times H$ into $M(H)$ is continuous,
(4) the mapping $(x, y) \mapsto \text{supp}(\delta_x * \delta_y) \in \mathcal{C}(H)$ is continuous with respect to the Michael topology on the space $\mathcal{C}(H)$ of nonvoid compact sets in H,
(5) there exists a unique $e \in H$ such that for every $x \in H$, $\delta_e * \delta_x = \delta_x * \delta_e = \delta_x$,
(6) there exists a necessarily unique involution (a homeomorphism $x \mapsto \tilde{x}$ of H onto itself with the property $\tilde{(\tilde{x})} = x$ for all $x \in H$) such that $\tilde{(\delta_x * \delta_y)} = \delta_{\tilde{y}} * \delta_{\tilde{x}}$,
(7) for $x, y \in H$, $e \in$supp$(\delta_x * \delta_y)$ if and only if $x = \tilde{y}$.

In the following we will write just H instead of $(M(H), +, *)$. It is still unknown if an arbitrary hypergroup admits a left Haar measure. It particular, it remains unknown whether every amenable hypergroup admits a left Haar measure. But all the known examples such as commutative hypergroups and central hypergroups do, for more information see [1] and [2]. In this case, one can define the convolution algebra $L^1(H)$ with multiplication $f * g(x) = \int f(x * y) g(\tilde{y}) d\lambda(y)$ for all $f, g \in L^1(H)$. Recall that $L^1(H)$ is a Banach subalgebra and an ideal in $M(H)$ with a bounded approximate identity [2]. It should be noted that these algebras include not only the group algebra $L^1(G)$ but also most of the semigroup algebras.

2. Main results

Let H be a hypergroup with left Haar measure λ. The first Arens product on $L^\infty(H)^*$ is defined in stages as follows.
Let $\mu, \nu \in L^1(H)$, $f \in L^\infty(H)$ and $F, G \in L^\infty(H)^*$;

(i) Define $f\mu \in L^\infty(H)$ by $\langle f\mu, \nu \rangle = \langle f, \mu * \nu \rangle$;
(ii) Define $Ff \in L^\infty(H)$ by $\langle Ff, \mu \rangle = \langle F, f\mu \rangle$;
(iii) Define $GF \in L^\infty(H)^*$ by $\langle GF, f \rangle = \langle G, Ff \rangle$.

$L^\infty(H)^*$ is a Banach algebra, for more details see [3].

Theorem 2.1. *Let H be a hypergroup with left Haar measure λ. Then the following conditions are equivalent:*

(i) *there exists $0 \neq \mu \in L^1(H)$ such that if $F \in L^\infty(H)^*$ and $\mu F \in L^1(H)$, then $F \in L^1(H)$;*
(ii) *H is discrete.*

The following corollary is a direct consequence of theorem 2.1.

Corollary 2.2. *Let H be a compact hypergroup. Then the following conditions are equivalent:*

(i) *there exists $0 \neq \mu \in L^1(H)$ such that if $F \in L^\infty(H)^*$ and $\mu F \in L^1(H)$, then $F \in L^1(H)$;*
(ii) *H is finite.*

Let H be a compact hypergroup. Let $\mu \in L^1(H)$. The mapping $x \mapsto \delta_x * \mu$ is weakly continuous. Since H is compact, $\{\delta_x * \mu;\ x \in H\}$ is relatively weakly compact. By the Krein-Smulian theorem the closed, convex, circled hull of $\{\delta_x * \mu; x \in H\}$ is also weakly compact. It follows that $\{\nu * \mu; \nu \in L^1(H), \|\nu\| \leq 1\}$ is relatively weakly compact. It is easy to see that $\{\mu F; F \in L^\infty(H)^*, \|F\| \leq 1\}$ is relatively weakly compact. Suppose that $F \in \{F \in L^\infty(H)^*; \|F\| \leq 1\}$ and $\{\nu_\alpha\}$ is a net in $\{\nu \in L^1(H); \|\nu\| \leq 1\}$ which converges to F in the weak*-topology. Therefore $\{\mu * \nu_\alpha\}$ converges to μF in the weak*-topology. Passing to a subnet if necessary, we can assume that $\{\mu * \nu_\alpha\}$ converges weak to a measure $\nu \in L^1(H)$. Consequently $\mu F = \nu \in L^1(H)$.
The next corollary is an immediate consequence of above explanation.

Corollary 2.3. *Let H be an infinite compact hypergroup. Then $L^1(H)$ is a right ideal in $L^\infty(H)$ and $L^1(H)$ is not reflexive.*

Theorem 2.4. *Let H be a hypergroup with left Haar measure λ. The following two properties of an element μ in $M(H)$ are equivalent:*

(i) *if $F \in L^\infty(H)^*$ and $\mu F \in L^1(H)$, then $F \in L^1(H)$;*
(ii) *if $\{\nu_n\}$ is a bounded sequence in $L^1(H)$ such that $\{\mu * \nu_n\}$ is weakly convergent, then $\{\nu_n\}$ contains a weakly convergent subsequence.*

For a non-empty subset S of $L^1(H)$. The annihilator of S, denoted $Ann(S)$, is the set of all elements ν in $L^1(H)$ such that, for all μ in S, $\mu * \nu = 0$. In set notation,

$$Ann(S) = \{\nu \in L^1(H); \mu * \nu = 0 \text{ for all } \mu \in S\}.$$

Proposition 2.5. *Let H be a hypergroup with left Haar measure λ. The following two properties of an element μ in $M(H)$ are equivalent:*

(i) *if $F \in L^\infty(H)^*$ and $\mu F \in L^1(H)$, then $F \in L^1(H)$;*
(ii) *$Ann(\mu)$ is reflexive and $\overline{\mu B_S} \subseteq \mu S$ for every closed subspace S of $L^1(H)$.*

Let $\mathbb{C}$ be the multiplicative group of all complex numbers. Let $\mu \in M(\mathbb{C})$. Consider the following assertions:

(i) if $\mu F \in L^1(\mathbb{C})$, then $F \in L^1(\mathbb{C})$;
(ii) $\nu \in M(\mathbb{C})$ and $\mu * \nu \in L^1(\mathbb{C})$ imply $\nu \in L^1(\mathbb{C})$.

Clearly (i) impies (ii). Are the converse implication true?

Proposition 2.6. *Assume that H is a commutative hypergroup. Let $\mu \in M(H)$, and let $\{\mu F; F \in L^\infty(H)^*\} + L^1(H)$ be a dense subspace of $L^\infty(H)^*$. If $\mu F \in L^1(H)$, then $F \in L^1(H)$.*

Proof. Let $F \in L^\infty(H)^*$ such that $\mu F \in L^1(H)$. Let $G \in L^\infty(H)^*$ and $\{\nu_\alpha\}$ be a net in $L^1(H)$ such that $\nu_\alpha \to G$ in the weak*-topology [3]. We can write

$$\mu F G = \lim_\alpha \mu F \nu_\alpha = \lim_\alpha \nu_\alpha * \mu F = G\mu F,$$

because H is commutative. This shows that $\mu FG = G\mu F$ for all $G \in L^\infty(H)^*$. Fix $G \in L^\infty(H)^*$. By assumption, $\{\mu F; F \in L^\infty(H)^*\} + L^1(H)$ is a dense subspace of $L^\infty(H)^*$. Consequently, we can find sequences $\{F_n\} \subseteq L^\infty(H)^*$ and $\{\mu_n\} \subseteq L^1(H)$ with $\{\mu F_n + \mu_n\}$ norm-convergent to G. Therefore

$$\begin{aligned} FG &= \lim_n F(\mu F_n + \mu_n) = \lim_n F\mu F_n + F\mu_n \\ &= \lim_n \mu F_n F + \mu_n F = \lim_n (\mu F_n + \mu_n)F = GF. \end{aligned}$$

Therefore $FG = GF$ for all $G \in L^\infty(H)^*$. We next show that $F \in Z_t(L^\infty(H)^*) = L^1(H)$ [2]. Indeed, if $\{G_\alpha\}$ is a net in $L^\infty(H)^*$ and $G_\alpha \to G$ in the weak*-topology, then

$$\begin{aligned} \lim_\alpha \langle FG_\alpha, f\rangle &= \lim_\alpha \langle G_\alpha F, f\rangle = \lim_\alpha \langle G_\alpha, Ff\rangle \\ &= \langle G, Ff\rangle = \langle GF, f\rangle, \end{aligned}$$

for all $f \in L^\infty(H)$. On the other hand, $\langle GF, f\rangle = \langle FG, f\rangle$. Hence $FG_\alpha \to FG$ (in the weak*-topology) implies that F is in the topological center of $L^\infty(H)^*$. This completes our proof. □

Recall that a basic sequence $\{x_n\}$ in a Banach space X is said to be boundedly complete if for each sequence of scalars $\{\alpha_n\}$, $\sum_{n=1}^\infty \alpha_n x_n$ is convergent whenever $\sup\{\|\sum_{i=1}^n \alpha_i x_i\|; n \in \mathbb{N}\} < \infty$.

Proposition 2.7. *Let H be a hypergroup with a left Haar measure, and let $\mu \in M(H)$. Consider the following assertions:*

(i) *If $\{\mu_n\}$ is a basic sequence in B and $\sum_{i=1}^\infty \|\mu * \mu_n\| < \infty$, then $\{\mu_n\}$ is boundedly complete;*

(ii) *$F \in L^\infty(H)^*$ and $\mu F \in L^1(H)$ imply $F \in L^1(H)$.*

Then the implication $(i) \to (ii)$ hold.

References

1. M. Amini, H. Nikpey and S.M. Tabatabaie, *Crossed product of C^*-algebras by hypergroups,* Mathematische Nachrichten, **292** (2019), 1897–1910.
2. W.R. Bloom and H. Heyer, *Harmonic analysis of probability measures on hypergroups*, vol. 20, de Gruyter Studies in Mathematics, Walter de Gruyter & Co., Berlin, 1995.
3. H.G. Dales, *Banach algebra and automatic continuity*, London Math. Soc. Monogr. Ser. Clarendon Press, 2000.
4. A. Ghaffari, T. Haddadi, S. Javadi and M. Sheybani, *On the structure of hypergroups with respect to the induced topology*, Rocky Mountain Journal of Mathematics, **52** (2022), 519–533.
5. B. Willson, *Configurations and invariant nets for amenable hypergroups and related algebras,* Transactions of the American Mathematical Society, **366** (2014), 5087–5112.