The Extended Abstracts of
The 24th Seminar on Mathematical Analysis and its Applications
26-27 May, 2021, Imam Khomeini International University, Qazvin, Iran

THE EQUIVALENCY OF P-BESSEL SEQUENCES WITH CONTROLLED P-BESSEL SEQUENCES

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Abstract. In this paper inspiring the concept of p-Bessel sequences, p-controlled Bessel sequences are presented and showed that in the case that $1 < p \leq 2$ these two concept could replace instead of each other.

1. Introduction

Controlled frames as a generalization of frames, have been introduced for getting an improved solution of a linear system of equation $Ax = B$, which this system can be solved by equation $PAx = PB$, where P is a suitable matrix to get a better duplicate algorithm [2]. Controlled frames for spherical wavelets were first introduced in [2] and the relation between controlled frames and standard frames were developed in [1, 4, 5, 6, 7].
In this paper, motivated the concept of p-frames, we introduce p-controlled frames on Banach spaces. In Section 2, we show that under some strong condition the concept of p-Bessel sequences and controlled p-Bessel sequences are equivalent. In other words, the equivalency of these two concepts presented when $1 < p \leq 2$, however the general case $1 < p < \infty$ is more desirable that we did not reach.
Throughout this paper $GL(X)$ is denoted as the set of all bounded and invertible operators on the space X.

1991 *Mathematics Subject Classification.* Primary 42C15; Secondary 42C40, 41A58.

Key words and phrases. p-frames; controlled p-frames; frame mappings.

A sequence $\{f_i\}_{i=1}^{\infty} \subseteq H$ is a frame for H if there exist $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{i=1}^{\infty} |\langle f, f_i \rangle|^2 \leq B\|f\|^2, \quad f \in H, \tag{1.1}$$

The constants A and B are called lower and upper frame bounds, respectively. The sequence $\{f_i\}_{i=1}^{\infty} \subseteq H$ is a Bessel sequence for H, if only the right hand inequality in (1.1) holds for all $f \in H$.
Let $\{f_i\}_{i=1}^{\infty}$ be a Bessel sequence for H. Then the operator

$$T : \ell^2 \to H, \quad T\{a_i\}_{i=1}^{\infty} = \sum_{i=1}^{\infty} a_i f_i,$$

is called the synthesis operator and its adjoint

$$T^* : H \to \ell^2, \quad T^* f = \{\langle f, f_i \rangle\}_{i=1}^{\infty},$$

is called the analysis operator of $\{f_i\}_{i=1}^{\infty}$.
Controlled frames with one and two controller operators were first introduced in [1] and [5], respectively. They are equivalent to standard frames and so this concept gives a generalization way to check the frame condition.

Definition 1.1. Let $\{f_i\}_{i=1}^{\infty}$ be a sequence of vectors in a Hilbert space $\mathcal{H}$ and $C, D \in GL(\mathcal{H})$. Then $\{f_i\}_{i=1}^{\infty}$ is called a frame controlled by C and D or (C, D)-controlled frame if there exist two constants $0 < A \leq B < \infty$, such that

$$A\|f\|^2 \leq \sum_{i=1}^{\infty} \langle f, Cf_i \rangle \langle Df_i, f \rangle \leq B\|f\|^2, \quad f \in \mathcal{H}. \tag{1.2}$$

If only the right inequality in (1.2) holds, then $\{f_i\}_{i=1}^{\infty}$ is called a (C, D)-controlled Bessel sequence. If $A = B$ then $\{f_i\}_{i=1}^{\infty}$ is called a (C, D)-controlled tight frame.

Let $F = \{f_i\}_{i=1}^{\infty}$ be a Bessel sequence of elements in H. The C-synthesis operator $T_{CF} : \ell^2 \to H$ is defined as

$$T_{CF}(\{a_i\}_{i=1}^{\infty}) = \sum_{i=1}^{\infty} a_i C f_i = CT(\{a_i\}_{i=1}^{\infty}), \quad \forall \{a_i\}_{i=1}^{\infty} \in \ell^2,$$

and the adjoint operator $T_{CF}^* : H \to \ell^2$ which is called the C-analysis operator is as follows:

$$T_{CF}^* f = \{\langle f, Cf_i \rangle\}_{i=1}^{\infty} = T^* C^* f.$$

Now we define the controlled frame operator S_{CD} on H

$$S_{CD}f = T_{DF}T^*_{CF}f = \sum_{i=1}^{\infty}\langle f, Cf_i\rangle Df_i, \quad \forall f \in H.$$

It is easy to see that if $F = \{f_i\}_{i=1}^{\infty}$ is a (C, D)-controlled frame with bounds A_{CD} and B_{CD}, then S_{CD} is well-defined and

$$A_{CD}Id_H \leq S_{CD} \leq B_{CD}Id_H.$$

Hence S_{CD} is a bounded, invertible, self-adjoint and positive linear operator. Therefore we have $S_{CD} = S^*_{CD} = S_{DC}$ [1, 5].

Definition 1.2. A sequence $\{g_i\}_{i=1}^{\infty} \subseteq X^*$ is a p-frame for X $(1 < p < \infty)$ if there exist constants, $0 < A \leq B < \infty$ such that

$$A\|f\| \leq (\sum_{i=1}^{\infty}|g_i(f)|^p)^{\frac{1}{p}} \leq B\|f\|. \tag{1.3}$$

The sequence $\{g_i\}_{i=1}^{\infty}$ is called a p-Bessel sequence if only the right inequality in (1.3) holds.

If $\{g_i\}_{i=1}^{\infty} \subseteq X^*$ is a p-frame for X. Then two operators

$$U : X \to \ell^p, \quad Uf = \{g_i(f)\}_{i=1}^{\infty},$$

and

$$T : \ell^q \to X^*, \quad T\{d_i\}_{i=1}^{\infty} = \sum_{i=1}^{\infty} d_i g_i,$$

is defined. The operator U is called the analysis operator and T is called the synthesis operator of $\{g_i\}_{i=1}^{\infty}$. If $\{g_i\}_{i=1}^{\infty}$ is a p-frame (or just a p-Bessel sequence) then U is a bounded operator and also T.

2. Main results

In this section first the concept of (C, D)-controlled p-Bessel sequence is introduced and then it is proved that under some suprising condition $1 < p \leq 2$ and $\overline{sgn(\langle f, Cg_i\rangle)}sgn(\langle f, Dg_i\rangle) > 0$ for each $f \in X$ and $i \in \mathbb{N}$, the concept of (C, D)-controlled p-Bessel sequence and p-Bessel sequence is equivalent.

Definition 2.1. Let $\{g_i\}_{i=1}^{\infty}$ be a family of vectors in X^*. Suppose that $C, D \in GL(X^*)$. The sequence $\{g_i\}_{i=1}^{\infty}$ is called a p-frame controlled by C and D or (C,D)-controlled p-frame if there exist constants $0 < A \leq B < \infty$ such that for each $f \in X$,

$$\begin{aligned} A^2\|f\|^2 &\leq \|\{\langle f, Cg_i\rangle\}_{i=1}^{\infty}\|_p^{2-p}\sum_{i=1}^{\infty}|\langle f, Dg_i\rangle||\langle Cg_i, f\rangle|^{p-1}\overline{sgn\langle f, Cg_i\rangle}sgn\langle f, Dg_i\rangle \\ &\leq B^2\|f\|^2. \end{aligned} \tag{2.1}$$

If only the right inequality in (2.1) satisfied, then $\{g_i\}_{i=1}^{\infty}$ is called a (C,D)-controlled p-Bessel sequence.

If $C = D = I_X$ in above definition then we see that $\{g_i\}_{i=1}^{\infty}$ is a p-frame for X.

Proposition 2.2. *Let $1 < p \leq 2$. Suppose that $\overline{sgn(\langle f, Cg_i\rangle)}sgn(\langle f, Dg_i\rangle) > 0$ for each $f \in X$ and $i \in \mathbb{N}$. Then the sequence $\{g_i\}_{i=1}^{\infty} \subseteq X^*$ is a (C,D)-controlled p-Bessel sequence if and only if it is a p-Bessel sequence.*

Proof. Suppose that $\{g_i\}_{i=1}^{\infty}$ is a (C,D)-controlled p-Bessel sequence. Then by (2.1)

$$(\sum_{i=1}^{\infty}|\langle Cg_i, f\rangle|^p)^{\frac{1}{p}} < \infty, \quad f \in X.$$

Suppose that there exists $0 \neq f_0 \in X$ such that for each $M > 0$

$$(\sum_{i=1}^{\infty}|\langle Cg_i, f_0\rangle|^p)^{\frac{1}{p}} > M. \tag{2.2}$$

Consider

$$\sum_{i=1}^{\infty}|\langle f_0, Dg_i|\rangle|\langle Cg_i, f_0\rangle|^{p-1}\overline{sgn\langle f_0, Cg_i\rangle}sgn\langle f_0, Dg_i\rangle = K.$$

Then three cases may happen:

(i) $K = \infty$.
(ii) $0 < K < \infty$.
(iii) $K = 0$.

Since $1 < p \leq 2$, the cases (i) and (ii) is a contradiction with (2.1) and (2.2).
(iii) If $K = 0$, then $\overline{sgn\langle f, Cg_i\rangle}sgn\langle f, Dg_i\rangle = 0$, which is a contradiction.
Therefore

$$(\sum_{i=1}^{\infty}|\langle Cg_i, f\rangle|^p)^{\frac{1}{p}} < \infty, \quad f \in X.$$

So similar to the proof of Lemma 3.1.1 in [3], there exists $B' > 0$ such that

$$(\sum_{i=1}^{\infty} |\langle g_i, f\rangle|^p)^{\frac{1}{p}} < B'\|f\|, \quad f \in X.$$

Now suppose that $\{g_i\}_{i=1}^{\infty}$ is a p-Bessel sequence with bound B for X. Since $1 < p \leq 2$, we have

$$\|\{\langle f, Cg_i\rangle\}_{i=1}^{\infty}\|_p^{2-p} \leq B^{2-p}\|C^* f\|^{2-p}, \quad f \in X. \tag{2.3}$$

Also since $\frac{1}{p}+\frac{1}{q} = 1$ and $\overline{sgn(\langle f, Cg_i\rangle)}sgn(\langle f, Dg_i\rangle) > 0$ for each $f \in X$ and $i \in \mathbb{N}$, we have

$$\begin{aligned}
\sum_{i=1}^{\infty} |\langle f, Dg_i\rangle||\langle Cg_i, f\rangle|^{p-1}\overline{sgn\langle f, Cg_i\rangle}sgn\langle f, Dg_i\rangle &= \sum_{i=1}^{\infty} |\langle f, Dg_i\rangle\langle f, Cg_i\rangle||\langle Cg_i, f\rangle|^{p-2} \\
&\leq \sum_{i=1}^{\infty} |\langle f, Dg_i\rangle||\langle Cg_i, f\rangle|^{p-1} \\
&\leq (\sum_{i=1}^{\infty} |\langle f, Dg_i\rangle|^p)^{\frac{1}{p}}(\sum_{i=1}^{\infty} |\langle Cg_i, f\rangle|^p)^{\frac{1}{q}} \\
&\leq B^p\|D^* f\|\|C^* f\|^{\frac{p}{q}}, \quad f \in X.
\end{aligned}$$

Therefore by (2.3) and above equations for each $f \in X$ we have

$$\begin{aligned}
&\|\{\langle f, Cg_i\rangle\}_{i=1}^{\infty}\|_p^{2-p} \sum_{i=1}^{\infty} |\langle f, Dg_i\rangle||\langle Cg_i, f\rangle|^{p-1}\overline{sgn(\langle f, Cg_i\rangle}sgn(\langle f, Dg_i\rangle) \\
&\qquad\qquad\qquad\qquad \leq B^2\|D\|\|C\|\|f\|^2.
\end{aligned}$$

So $\{g_i\}_{i=1}^{\infty}$ is a (C, D)-controlled p-Bessel sequence. □

3. Further remarks

The equivalency of p-Bessel sequences with (C, D)-controlled p-Bessel sequences is our aim in this paper that we just obtain it in the case of $1 < p \leq 2$, however the general case $1 < p < \infty$ is more desirable that we did not reach and it remains as an open question.

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