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REDUCIBLE M-IDEALS IN BANACH SPACES

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Abstract. For an arbitrary nontrivial M-ideal J in a Banach space X, is there a M-ideal K in X such that $J \cap K$ is reducible? To answer this question, we look for conditions under which the set

$R_J - M(X) := \{K \in N-M-ideal(X) : J \cap K \in R-M-ideal(X)\}$

is nonempty, where $M - ideal(X)$ and $R - M - ideal(X)$ denote the set of of M-ideals and reducible M-ideals in X, respectively. Further, some important properties on $R_J - M(X)$ are established.

1. Introduction

The concept of an M-ideal was introduced by Alfsen and Effros in [1]. In the theory of Banach spaces, M-ideals are an important tool to study geometric and isometric properties of the spaces. A closed subspace J of a Banach space X is called M-ideal if there exist a linear projection $P : X^* \to J^\perp$ such that

$$\|f\| = \|Pf\| + \|f - Pf\|$$

for all $f \in X^*$, where $J^\perp = \{f \in X^* : f(x) = 0 \ for \ any \ x \in J\}$ is the annihilator of J in X^*. 0 and X are called trivial M-ideals and all

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the other M-ideals will be called nontrivial. Some authors have studied M-ideals via intersection property of balls. (See [4],[5])

An M-ideal generalizes the two-sided ideals in a C^*-algebra; Indeed, an M-ideal in the self-adjoint part of a C^*-algebra is exactly the self-adjoint part of a closed two-sided ideal. According to the geometric characterization of the ideals in C^*-algebras, the M-ideals have been identified with the two sided ideals [9]. The notion of an ideal was introduced by Godefroy, Kalton and Saphar in [2]. A closed subspace Y of a Banach space X is said to be an ideal in X if $Y^\perp$ (the annihilator of Y in the dual space X^* of X) is the kernel of a projection of norm one in X. (See[8] for more details)

There are a number of important properties shared by M-ideals, but not by arbitrary subspaces. For example, M-ideals have property U, i.e., every norm on linear functional on an M-ideal has a unique norm one extension to phase space ([3], [7]). Authors in [7] have introduced property $(+U)$: A closed subspace Y of a Banach space X has property $(+U)$ in X, when it has property U in both X and Y^{**}, the second dual space of Y. They proved that if X is an M-embedded space, then every M-ideal in X has property $(+U)$ in it. For more details on the topic of M-ideals and their properties, the reader is referred to [1], [3], [9], [10], [11].

Reducible M-ideals are a subclass of M-ideals. Uttersrud in [10], has mentioned that the definition of a reducible M-ideal is due to Alfsen. An M-ideal J in a Banach space X is reducible if there exist M-ideals J_1 and J_2, $J \neq J_1, J_2$ such that $J = J_1 \cap J_2$. An M-ideal is irreducible M-ideal J is hyperplane, then X is isometric to a L^1-predual space([11]). A Banach space X is said to be an L^1-predual provided its dual X^* is isometric to $L^1(\mu)$ for some measure space (Ω, Σ, μ). A subspace M is hyperplane if it has codimension 1, i.e., $codim(M) = dim(X \diagup M) = 1$.

The subject of reducible M-idealin Banach spaces seems to be a little-understood area. In other words, little research has been done on reducible M-ideals. Therefore, new results can be achieved in this regard. Authors in [6] have obtained some results on reducible M-ideals in Banach spaces. For instance, they determined the general form of a reducible M-ideal in the space of continuous functions on a locally compact space. They also introduced the concept of a semi reducible M-ideal in a Banach space. An M-ideal J in a Banach space X is semi reducible if there exit an M-ideal K and a closed subspace F such that $K, F \neq J$ and $J = K \cap F$.

The main goal of our presented research paper is to give more details on reducible M-ideals. Throughout this paper, X is a real Banach space. $L(X)$ and $K(X)$ denote the spaces of linear continuous and linear compact operators, respectively. By $M-ideal(X)$ and $R-M-ideal(X)$ we mean the set of all M-ideals and reducible M-ideals in a Banach space X, respectively. Also $N-M-ideal(X)$ denotes the set of nontrivial M-ideals in X. Any unexplained notion can be found in [3] and [6].

2. Main results

For $J \in M-ideal(X)$, we define

$$R_J - M(X) := \{K \in N-M-ideal(X) : J \cap K \in R-M-ideal(X)\}$$

and

$$I-R_J-M(X) := \{K \in N-M-ideal(X) : J \cap K \notin R-M-ideal(X)\}.$$

Notice that $J \in R_J - M(X)$ if and only if J is reducible. Further, $R_X - M(X) = R - M - ideal(X)$. Also, $R_0 - M(X) = \emptyset$ if and only if $0 \notin R - M - ideal(X)$.

Next we show that $I - R_J - M(X) \neq \emptyset$, for every $J \in N - M - ideal(X)$ whereas $R_J - M(X)$ is not necessarily nonempty.

For an M-ideal J, we use $M_J(X)$ to denote the set of nontrivial M-ideals contained in J ([6]). A nontrivial M-ideal J in X is called to be maximal M-ideal if when K is an nontrivial M-ideal in X containing J, then $K = J$ ([6]). Clearly, a maximal M-ideal is not reducible. By $Max - M - ideal(X)$, we mean the set of all nontrivial maximal M-ideals in X.

Proposition 2.1. *Let $J \in N-M-ideal(X)$. Then $I-R_J-M(X) \neq \emptyset$.*

Proof. Arguing by contradiction, we assume that $I - R_J - M(X) = \emptyset$. Thus J is reducible in X. Further, it follows from $I - R_J - M(X) = \emptyset$ that $J \notin M_K(X)$ and $K \notin M_J(X)$, for every nontrivial M-ideal K. This implies that J is a maximal M-ideal in$X$. This contradiction completes the proof.

Corollary 2.2. *There is no nontrivial M-ideal J in a Banach space for which $R_J - M(X) = N - M - ideal(X)$.*

$R_J - M(X)$ need not be always nonempty; for instance if $1 < p < \infty$, $X = L(l_p)$ and $J = K(l_p)$ then $R_J - M(X) = \emptyset$ ([10] , [6]).

In the following, we give a sufficient condition for $R_J - M(X)$ to be nonempty. For a real-valued continuous function f on X (i.e.,$f \in C(X)$) and a subspace Y of X, $f_{|Y}$ denotes the restriction of f to Y.

Theorem 2.3. *Let J be a maximal M-ideal in X. Suppose further, if $f \in C(X))$ vanishes on every nontrivial M-ideal in X, then $f \equiv 0$ on X. Then $R_J - M(X) \neq \emptyset$.*

Proof. Assume by contradiction that $R_J - M(X) = \emptyset$. Then $J \in I - R_K - M(X)$ for every $K \in N - M - ideal(X)$. We have two cases:

Case1 . $J \cap K = J$ which contradicts to maximality of J.
Case2 .$J \cap K = K$ for every $K \in N - M - ideal(X)$.

Let $x_0 \in X \diagdown J$ and take $Y := \{x_0\}$. Now Urysons lemma infers that there exists $f \in C(X))$ such that and ([12, p.102]). Since, in this case, nontrival M-ideals are contained in J, we get a contraction. □

Remark 2.4. There are another sufficient conditions for $R_J - M(X)$ to be nonempty; For example, if J is a maximal M-ideal and the set $Max - M - ideal(X)$ is not singleton, then $R_J - M(X) \neq \emptyset$. Also, if $J \in Max - M - ideal(X)$ and there exist a nontrival M-ideal K with $K \notin M_J(X)$, then $R_J - M(X) \neq \emptyset$.

Proposition 2.5.

(i) *Suppose that $J \subset K$ and $L \in M_J(X)$, for every nontrivial M-ideal L in K. Then $R_J - M(X) \subset R_K - M(X)$.*
(ii) *If $J \in Max - M - ideal(X)$,then $I - R_J - M(X) = M_J(X)$.*
(iii) $R - M - ideal(X) \subset \bigcup_{J \in N-M-ideal(X)} R_J - M(X)$.
(iv) *Suppose that $J \in M_K(X)$ and $K \in M_J(X)$. Then $R_J - M(X) \neq \emptyset$ if and only if $R_k - M(X) \neq \emptyset$.*

Proof. (i) Suppose that $L \in R_J - M(X)$. Assume for a contradiction that $L \notin R_K - M(X)$. Then $L \cap K$ is not reducible. Therefore, $L \subset K$ or $K \subset L$. If $L \subset K$, one can conclude from assumption that $L \subset J$. Thus $L \cap J$ is not reducible which is a contradiction. If $K \subset L$, we get a similar contradiction.
(ii) Use Proposition2.1 as well as the maximality of J.
(iii) Straightforward.
(iv) It is similar to the one used to prove (i). □

In [6], Khorshidvandpour and Aminpour constructed a process called R-process: if $J \in M - ideal(X)$ and there exist a nontrival $M - ideal K$ in X containing J, then we remove J and repeat the process for K. Continuing this process gives us a subset of M-ideals in X which denoted by $M - M - ideal(X)$. It follows from $K \in I - R_J - M(X)$ that $K \subset J$ or $J \subset K$. Therefore, the concept of $I - R_J - M(X)$ may be an useful tool in R-process.

If $|M-ideal(X)|<\infty$*, the inequality* $|R-M-ideal(X)|<|M-ideal(X)|$ *is proper; whereas the equality can be happen in the infinite case. Moreover, if* $|M-ideal(X)|<\infty$*, it is possible* $R-M-ideal(X)=\emptyset$ *and* $M-M-ideal(X)\neq\emptyset$ *(for example, if* $1<P<\infty$*,* $X=L(l_p)$ *then* $R-M-ideal(X)=\emptyset$ *and* $M-M-ideal(X)=\{K(l_p)\}$ *([10] , [6]); but it is impossible in the infinite case. Observe the following result.*

Proposition 2.6. *Let* $M-ideal(X)$ *be an infinite set. If* X *has no reducible M-ideal, then* $M-M-ideal(X)=\emptyset$*.*

Proof. We prove the assertion when $M-ideal(X)$ is countably infinite. The proof of uncountable case is similar. We take $M-ideal(X)=\{J_1,J_2,...\}$. Since $R-M-ideal(X)=\emptyset$, we have $J_i\cap J_j\notin R-M-ideal(X)$ for every $i\neq j$. Hence for every $J_i,J_j\in M-ideal(X)$ we have $J_i\subset J_j$ or $J_j\subset J_i$. This means that $M-ideal(X)$ is a totally ordered set by set inclusion. Therefore, by a suitable rearrangement of indices one can write that $M-ideal(X)=\{J_{\alpha_1},J_{\alpha_2},...\}$ such that $J_{\alpha_1}\subset J_{\alpha_2}\subset ...$, where $\alpha_1<\alpha_2<....$ It follows that $M-M-ideal(X)=\emptyset$. □

Corollary 2.7. *If* $M-ideal(X)$ *is an infinite set and* $M-M-ideal(X)$ *is nonempty, then* $R-M-ideal(X)\neq\emptyset$*.*

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