**Reducible M-ideals in Banach Spaces**

Sajad Khorshidvandpour 1,\*and Parastoo Heiatian Naeini 2

**Abstract.** For an arbitrary nontrivial M-ideal in a Banach space , is there an M-ideal in such that is reducible? To answer this question, we look for conditions in which the set

is nonempty, where and denote the set of M-ideals and reducible M-ideals in , respectively. Further, some important properties on are established.

**Key words and phrases**: M-ideal, Reducible M-ideal, Maximal M-ideal.

**2010 Mathematics Subject Classification**: 46B20, 46B25.

**1. Introduction**

The concept of an M-ideal was introduced by Alfsen and Effros in [1]. In the theory of Banach spaces, M-ideals are an important tool to study geometric and isometric properties of the spaces. A closed subspace of a Banach space is called -ideal if there exist a linear projection such that

for all , where is the annihilator of in . and are called trivial -ideals and all the other -ideals will be called nontrivial. Some authors have studied M-ideals via intersection property of balls. (See [4], [5])

An -ideal generalizes the two-sided ideals in a \*-algebra; Indeed, an M-ideal in the self-adjoint part of a -algebra is exactly the self-adjoint part of a closed two-sided ideal. According to the geometric characterization of the ideals in \*-algebras, the -ideals have been identified with the two sided ideals [9]. The notion of an ideal was introduced by Godefroy, Kalton and Saphar in [2]. A closed subspace of a Banach space X is said to be an ideal in X if (the annihilator of Y in the dual space of X) is the kernel of a projection of norm one in X. (See [8] for more details)

There are a number of important properties shared by -ideals, but not by arbitrary subspaces. For example, -ideals have property , i.e., every norm one linear functional on an -ideal has a unique norm one extension to phase space ([3], [7]) . Authors in [7] have introduced property A closed subspace of a Banach space has property in , when it has property in both and , the second dual space of . They proved that if is an M-embedded space, then every M-ideal in has property in it. For more details on the topic of -ideals and their properties, the reader is referred to [1], [3], [9], [10], [11].

Reducible -ideals are a subclass of -ideals. Uttersrud in [11], has mentioned that the definition of a reducible -ideal is due to Alfsen. An -ideal in a Banach space is reducible if there exist -ideals and , , such that . An -ideal is irreducible if it is not reducible. If a Banach space is such that every irreducible M-ideal is a hyperplane, then is isometric to a -predual space([11]). A Banach space is said to be an-predual provided its dual is isometric to for some measure space*.* A subspace M is a hyperplane if it has codimension 1, i.e., .

The subject of reducible M-idealin Banach spaces seems to be a little-understood area. In other words, little research has been done on reducible M-ideals. Therefore, new results can be achieved in this regard. Authors in [6] have obtained some results on reducible M-ideals in Banach spaces. For instance, they determined the general form of a reducible M-ideal in the space of continuous functions on a locally compact space. They also introduced the concept of a semi reducible M-ideal in a Banach space. An M-ideal in a Banach space is semi reducible if there exit an M-ideal and a closed subspace such that and .

The main goal of our presented research paper is to give more details on reducible -ideals. Throughout this paper, is a real Banach space. and denote the spaces of linear continuous and linear compact operators, respectively. By and we mean the set of all -ideals and reducible -ideals in a Banach space , respectively. Also denotes the set of nontrivial -ideals in . Any unexplained notion can be found in [3] and [6].

**2. Main Results**

For , we define

and

.

Notice that if and only if is reducible. Further, . Also, if and only if .

Next we show that , for every whereas is not necessarily nonempty.

For an M-ideal , we use to denote the set of nontrivial M-ideals contained in ([6]). A nontrivial -ideal in is called to be maximal -ideal if when is an nontrivial -ideal in containing , then ([6]). Clearly, a maximal -ideal is not reducible. By , we mean the set of all nontrivial maximal -ideals in .

**Proposition 2.1.** Let . Then .

**Proof.** Arguing by contradiction, we assume that . Thus is reducible in . Further, it follows from that and , for every nontrivial M-ideal . This implies that is a maximal -ideal in . This contradiction completes the proof. □

**Corollary 2.2.** There is no nontrivial M-ideal in a Banach space for which

need not be always nonempty; for instance if , and then ([10] , [6, Proposition2.2(ii)]).

In the following, we give a sufficient condition for to be nonempty. For a real-valued continuous function on and a subspace of , denotes the restriction of to .

**Theorem2.3.** Let be a maximal M-ideal in .Suppose further, if vanishes on every nontrivial -ideal in , then on . Then .

**Proof.** Assume by contradiction that . Then for every . We have two cases:

Case1. which contradicts to maximality of .

Case2. for every .

Let and take . Now, Uryson’s lemma infers that there exists such that and ([12, p.102]). Since, in this case, nontrivial -ideals are contained in , we get a contradiction □

**Remark 2.4.** There are another sufficient conditions for to be nonempty; For example, if is a maximal M-ideal and the set is not singleton, then . Also, if and there exist a nontrivial M-ideal with , then .

Let us prove some results on and .

**Proposition2.5.**

(i)Suppose that and , for every nontrivial M-ideal in . Then .

(ii) If ,then .

(iii)

(iv)Suppose that and . Then if and only if

**Proof.** (i) Suppose that . Assume for a contradiction that . Then is not reducible. Therefore, or . If , one can conclude from assumption that . Thus is not reducible which is a contradiction. If , we get a similar contradiction.

(ii)Use Proposition2.1 as well as the maximality of .

(iii)Straightforward.

(iv) It is similar to the one used to prove (i). □

In [6], Khorshidvandpour and Aminpour constructed a process called -process: if and there exist a nontrivial in containing , then we remove and repeat the process for . Continuing this process gives us a subset of M-ideals in which denoted by . It follows from that or . Therefore, the concept of may be an useful tool in R-process.

If , the inequality is proper; whereas the equality can be happen in the infinite case. Moreover, if , it is possible and (for example, if , then and ([10] , [6, Proposition2.2(ii)])); but it is impossible in the infinite case. Observe the following result.

**Proposition 2.6.** Let be an infinite set. If has no reducible M-ideal, then .

**Proof.** We prove the assertion when is countably infinite. The proof of uncountable case is similar. We take . Since , we have for every , . Hence for every we have . This means that is a totally ordered set by set inclusion. Therefore, by a suitable rearrangement of indices one can write that such that , where . It follows that . □

**Corollary 2.7.** If is an infinite set and is nonempty, then .

**References**

[1] E.M.Alfsen and E.G.Effros, Structure in real Banach space, Part I and II, Ann. of Math., 96 (1972), 98-173.

[2] G. Godefroy, N.J. Kalton and P. Saphar, Unconditional ideals in Banach spaces, Studia Math. 104 (1993), 13-59.

[3] P.Harmand , D.Werner and W.Werner, -ideals in Banach spaces and Banach algebras,

Lecture notes in Mathematics, vol. 1547, Springer,,Berlin,1993.

[4] C.R.Jayanarayanan , Intersection Properties of Balls in Banach Spaces, **Journal of Function Spaces and Applications, 2013, 1-10.**

[5] C.R.Jayanarayanan and T. Paul, rayanan Strong proximinality and intersection properties of balls in Banach spaces, Journal of Mathematical Analysis and Applications, 426(2015), 1217-1231.

[6] S.Khorshidvandpour and A.M.Aminpour, On the Reducible -ideals in Banach Spaces, Sahand Communication in Mathematical Analysis, 7(1) (2017), 27-37.

[7] S.Khorshidvandpour and A.M.Aminpour, Property and Strongly Smoothly Embedded Subspaces of a Banach Space, Advances and Applications in Mathematical Sciences, 16(8) (2017), 259-274.

[8] T.SS.R.K RAO, ON IDEALS IN BANACH SPACES, Rocky mountain journal of mathematics, 31(2) (2001), 595-609.

[9] R. R. Smith and J. D. Ward, -ideal structure in Banach algebras, J. Functional Analysis., 27 (1978), 337-349.

[10] R. R. Smith and J. D. Ward, -ideals in , Pacific Journal of Mathematics., 81(1) (1979), 227-237.

[11] U. Uttersrud, On -ideals and the Alfsen-Effros structure topology, Math. Scand., 43 (1978), 369-381.

[12] S.Willard, General Topology, Addison-Wesley, Reading, MA, 1970.

1Department of Mathematics, Faculty of Mathematical Sciences and Computer, Shahid Chamran University of Ahvaz, Ahvaz, Iran

2Department of Mathematics, Payame Noor University , Naein, Iran

\*Corresponding author, Email-Address: [skhorshidvandpour@gmail.com](mailto:skhorshidvandpour@gmail.com)