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A NOTE ON MULTIPLICATIVELY LOCAL SPECTRAL SUBSPACE PRESERVING MAPS

R. PARVINIANZADEH

Department of Mathematics, University of Yasouj, Yasouj 75918, Iran
r.parvinian@yu.ac.ir

Abstract. Let $B(X)$ be the algebra of all bounded linear operators on infinite-dimensional complex Banach space X. For $T \in B(X)$ and $\lambda \in \mathbb{C}$, let $X_T(\{\lambda\})$ denotes the local spectral subspace of T associated with $\{\lambda\}$. We investigate the form of all maps φ_1 and φ_2 on $B(X)$ such that, for every T and S in $B(X)$, the local spectral subspace of TS and $\varphi_1(T)\varphi_2(S)$ are the same associated with singleton set $\{\lambda\}$. Also, we obtain some interesting results in direction when $X = \mathbb{C}^n$.

1. Introduction

Throughout this paper, Let $B(X)$ be the algebra of all bounded linear operators on infinite-dimensional complex Banach space X and its unit will be denoted by I. For any vector $x_0 \in X$, let $B_{x_0}(X)$ be the collection of all operators in $B(X)$ vanishing at x_0. The local resolvent set, $\rho_T(x)$, of an operator $T \in B(X)$ at some point $x \in X$ is the set of all $\lambda \in \mathbb{C}$ for which there exists an open neighborhood U of λ in $\mathbb{C}$ and a X-valued analytic function $f : U \longrightarrow X$ such that $(\mu I - T)f(\mu) = x$ for all $\mu \in U$. The complement of local resolvent set is called the local spectrum of T at x, denoted by $\sigma_T(x)$, and is obviously a closed subset

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(possibly empty) of $\sigma(T)$, the spectrum of T. For every subset $F \subseteq \mathbb{C}$ the local spectral subspace $X_T(F)$ is defined by

$$X_T(F) = \{x \in X : \sigma_T(x) \subseteq F\}.$$

Clearly, if $F_1 \subseteq F_2$ then $X_T(F_1) \subseteq X_T(F_2)$. For more information about these notions one can see the book [1].

The problem of describing linear or additive maps on $B(X)$ preserving the local spectra has been initiated by Bourhim and Ransford in [5], and continued by several authors; see for instance [3] and the references therein. Motivated by the result from the theory of linear preservers proved by Jafarian and Sourour [7], Dolinar et al. [6], characterised the form of maps preserving the lattice of sum of operators, they showed that maps (not necessarily linear) $\varphi : B(X) \to B(X)$ satisfy $Lat(\varphi(A) + \varphi(B)) = Lat(A + B)$ for all $A, B \in B(X)$, if and only if there is a non zero scalar α and a map $\phi : B(X) \to K$ such that $\varphi(A) = \alpha A + \phi(A)I$ for all $A \in B(X)$. Recall that $X_T(\Omega)$, the local spectral subspace of T associated with a subset Ω of $\mathbb{C}$, is an element of $Lat(T)$, so one can replace the lattice preserving property by the local spectral subspace preserving property.
In [2], Benbouziane et al. characterized the forms of all maps preserving the local spectral subspace of sum, difference, product and triple product of operators associated with a singleton.
For a vector $x \in X$ and a linear functional f in the dual space X^* of X, let $x \otimes f$ stands for the operator of rank at most one defined by

$$(x \otimes f)y = f(y)x, \qquad \forall\ y \in X.$$

We denote $F_1(X)$ the set of all rank-one operators on X and $N_1(X)$ be the set of nilpotent operators in $F_1(X)$. Note that $x \otimes f \in N_1(X)$ if and only if $f(x) = 0$.

The following lemma gives an explicit identification of local spectral subspace in the case of rank-one operator.

Lemma 1.1. [5] *Let $R \in F_1(X)$ be a non-nilpotent operator, and let λ be a nonzero eigenvalue of R. Then $X_R(0) = ker(R)$ and $X_R(\{\lambda\}) = Im(R)$.*

The nonzero local spectrum of $T \in B(X)$ at any $x_0 \in X$ is defined by

$$\sigma_T^*(x_0) := \begin{cases} \{0\} & if\ \sigma_T(x_0) = \{0\}, \\ \sigma_T(T) \setminus \{0\} & if\ \sigma_T(x_0) \neq \{0\}. \end{cases}$$

Lemma 1.2. [4] *For a nonzero vector $x_0 \in X$ and a nonzero operator $R \in B(X)$, the following statements are equivalent.*

(a) R has rank one.
(b) $\sigma^*_{RT}(x_0)$ contains at most one element for all $T \in B(X)$.

In this paper, we investigate the form of all maps φ_1 and φ_2 on $B(X)$ such that, for every T and S in $B(X)$, the local spectral subspace of TS and $\varphi_1(T)\varphi_2(S)$ are the same associated with the singleton set $\{\lambda\}$.

2. Main results

The following Lemma is a key of the proofs coming after.

Lemma 2.1. [2] *Let x be a nonzero vector in X and $T, S \in B(X)$. If $X_T(\{\lambda\}) = X_S(\{\lambda\})$ for all $\lambda \in \mathbb{C}$. Then, $\sigma_T(x) = \{\mu\}$ if and only if $\sigma_S(x) = \{\mu\}$ for all $\mu \in \mathbb{C}$.*

This theorem will be useful in the proofs of the main results.

Theorem 2.2. [2] *Let $T, S \in B(X)$. The following statements are equivalent.*
(1) $T = S$
(2) $X_{TR}(\{\lambda\}) = X_{SR}(\{\lambda\})$ *for all* $\lambda \in \mathbb{C}$ *and* $R \in F_1(X)$.

Theorem 2.3. *If two surjective linear maps φ_1 and φ_2 from $B(X)$ onto $B(X)$ satisfy*

$$X_{\varphi_1(T)\varphi_2(S)}(\{\lambda\}) = X_{TS}(\{\lambda\}), \quad \forall\ T, S \in B(X),\ \forall\ \lambda \in \mathbb{C}$$

then φ_2 maps $B_{x_0}(X)$ onto $B_{x_0}(X)$ and there exist two bijective linear mappings $A : X \to X$ and $B : X \to X$ such that

$$\varphi_1(T) = ATB, \quad (T \in B(X)),$$

and

$$\varphi_2(T) = B^{-1}TA^{-1}, \quad (T \notin B_{x_0}(X)).$$

Proof. We break down the proof of Theorem into several steps.
Step 1. φ_1 is bijective.
Step 2. φ_1 preserves rank one operators in both directions.
step 3. $\varphi_2(B_{x_0}(X)) = B_{x_0}(X)$.
Step 4. There are bijective linear mappings $P : X \to X$ and $Q : X^\star \to X^\star$ such that $\varphi_1(x \otimes f) = Px \otimes Qf$ for all $x \in X$ and $f \in X^\star$.
Step 5. For any $x \in X$ and $f \in X^\star$, we have $f(x) = (Qf)(\varphi_2(I)Px)$
Step 6. P is continuous and $\varphi_2(I)$ is invertible.
Step 7. $\varphi_2(T) = B^{-1}TA^{-1}$ for all $T \notin B_{x_0}(X))$, where $A = \alpha^{-1}P$ for some nonzero scalar $\alpha \in \mathbb{C}$ and $B = (\varphi_2(I)A)^{-1}$.
Step 8. $\varphi_1(T) = ATB$ for every $T \in B(X)$. □

In the case X is a finite dimensional space, we have a good description of the concepts involved in local spectral theory; see for instance [8]. Let $M_n(\mathbb{C})$ denote the algebra of all $n \times n$ complex matrices, and for any vector $x_0 \in \mathbb{C}^n$, let $M_{n,x_0}(\mathbb{C})$ be the collection of all matrices in $M_n(\mathbb{C})$ vanishing at x_0.

Remark. [8]. Let $T \in M_n(\mathbb{C})$ and $\lambda_1, \lambda_2, ..., \lambda_r$ be the distinct eigenvalues of T and denote by $E_1, E_2, ..., E_r$ the corresponding root spaces. We have $\mathbb{C}^n = E_1 \oplus E_2 \oplus ... \oplus E_r$ and $T = T_1 \oplus T_2 \oplus ... \oplus T_r$ where T_i is the restriction of T to E_i. It follows that for every $x \in \mathbb{C}^n$,

$$\sigma_T(x) = \bigcup \{\sigma_{T_i}(P_i x) : \ 1 \leq i \leq r\} = \{\lambda_i : \ 1 \leq i \leq r, P_i(x) \neq 0\}$$

where $P_i : \mathbb{C}^n \to E_i$ is the canonical projection.

However, if $X = \mathbb{C}^n$, then the surjectivity φ_1 and φ_2 in Theorem 2.3 is redundant, as is shown by our next result.

Theorem 2.4. *Two maps φ_1 and φ_2 on $M_n(\mathbb{C})$ satisfy*

$$X_{\varphi_1(T)\varphi_2(S)}(\{\lambda\}) = X_{TS}(\{\lambda\}) \quad \forall\ T, S \in M_n(\mathbb{C}), \quad \forall\ \lambda \in \mathbb{C}$$

if and only if φ_2 maps $M_{n,x_0}(\mathbb{C})$ into itself and there are two invertible matrices A and B in $\mathbb{C}^n$ such that

$$\varphi_1(T) = ATB, \quad (T \in M_n(\mathbb{C})),$$

and

$$\varphi_2(T) = B^{-1}TA^{-1}, \quad (T \notin M_{n,x_0}(\mathbb{C})).$$

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