



ON MAPS PRESERVING OPERATORS OF LOCAL SPECTRAL SUBSPACE ASSOCIATED WITH ZERO

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ABSTRACT. Let $B(X)$ be the algebra of all bounded linear operators on Banach space X . For $T \in B(X)$, let $X_T(\{0\})$ denotes the local spectral subspace of T associated with $\{0\}$. We describe surjective linear maps φ on $B(X)$ that satisfy

$$X_{\varphi(T)}(\{0\}) = X_T(\{0\}), \quad \forall T \in Q(X).$$

Furthermore, we characterize maps φ (not necessarily linear nor surjective) on $B(X)$ which satisfy $X_{\varphi(T)-\varphi(S)}(\{0\}) = X_{T-S}(\{0\})$ for every $T, S \in B(X)$ which $T - S \in Q(X)$.

1. INTRODUCTION

Linear preserver problems, in the most general setting, demand the characterization of linear maps between algebras that leave a certain property, a particular relation, or even a subset invariant. This subject is very old and goes back well over a century to the so-called first linear preserver problem, due to Frobenius [6], that determines linear maps preserving the determinant of matrices. Given a Banach space X over the complex field \mathbb{C} , we shall denote by $B(X)$ the algebra of all linear bounded operators on X and its unit will be denoted by I . The local resolvent set, $\rho_T(x)$, of an operator $T \in B(X)$ at some point $x \in X$ is the set of all $\lambda \in \mathbb{C}$ for which there exists an open neighborhood

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U of λ in \mathbb{C} and a X -valued analytic function $f : U \rightarrow X$ such that $(\mu I - T)f(\mu) = x$ for all $\mu \in U$. The complement of local resolvent set is called the local spectrum of T at x , denoted by $\sigma_T(x)$, and is obviously a closed subset (possibly empty) of $\sigma(T)$, the spectrum of T . The local spectral radius of T at x is given by $r_T(x) := \limsup_{n \rightarrow \infty} \|T^n(x)\|^{\frac{1}{n}}$, and coincides with the maximum modulus of $\sigma_T(x)$ provided that T has the single-valued extension property. We recall an operator $T \in B(X)$ is said to have the single-valued extension property (henceforth abbreviated to SVEP) provided that for every open subset U of \mathbb{C} , the equation

$$(\mu I - T)f(\mu) = 0, \quad \forall \mu \in U,$$

has no nontrivial analytic solution f . The set of all operators in $B(X)$ which have the single-valued extension property will be denoted by $Q(X)$. It is easily verified that $T \in B(X)$ has the single-valued extension property if no nonempty open subset of \mathbb{C} is contained in the point spectrum of T . In particular, $T \in Q(X)$ if T has no eigenvalues or if the spectrum of T is nowhere dense in \mathbb{C} . The notion of SVEP at a point dates back to Finch [5]. For every subset $F \subseteq \mathbb{C}$ the local spectral subspace $X_T(\Omega)$ is defined by

$$X_T(\Omega) = \{x \in X : \sigma_T(x) \subseteq \Omega\}.$$

Clearly, if $\Omega_1 \subseteq \Omega_2$ then $X_T(\Omega_1) \subseteq X_T(\Omega_2)$. For more information about these notions one can see the books [1], [7].

Recently, there has been an upsurge of interest in linear and nonlinear local spectra preserver problems, which demand the characterization of maps on matrices or Banach space operators that leave the local spectra invariant. Bourhim and Ransford were the first ones to consider this type of preserver problem, characterizing in [4] additive maps on the algebra of all linear bounded operators on a complex Banach space X that preserve the local spectrum of operators at each vector of X . Their results cleared the way for several authors to describe maps on matrices or operators that preserve local spectrum and local spectral radius; see, for instance, the last section of the survey article [3] and the references therein.

For a vector $x \in X$ and a linear functional f in the dual space X^* of X , let $x \otimes f$ stands for the operator of rank at most one defined by

$$(x \otimes f)y = f(y)x, \quad \forall y \in X.$$

We denote $F_1(X)$ the set of all rank-one operators on X and $N_1(X)$ be the set of nilpotent operators in $F_1(X)$. Note that $x \otimes f \in N_1(X)$

if and only if $f(x) = 0$.

The first lemma summarizes some known basic properties of the local spectrum.

Lemma 1.1. [1], [7] *Let X be a Banach space and $T \in B(X)$. For every $x, y \in X$ and a scalar $\alpha \in \mathbb{C}$ the following statements hold.*

- (a) *If T has SVEP, then $\sigma_T(x) \neq \emptyset$ provided that $x \neq 0$.*
- (b) *$\sigma_T(\alpha x) = \sigma_T(x)$ if $\alpha \neq 0$, and $\sigma_{\alpha T}(x) = \alpha \sigma_T(x)$.*
- (c) *If $Tx = \lambda x$ for some $\lambda \in \mathbb{C}$, then $\sigma_T(x) \subseteq \{\lambda\}$. If, further, $x \neq 0$ and T has SVEP, then $\sigma_T(x) = \{\lambda\}$.*
- (d) *If $S \in B(X)$ commutes with T , then $\sigma_T(Sx) \subseteq \sigma_T(x)$.*
- (e) *$\sigma_{T^n}(x) = \{\sigma_T(x)\}^n$ for all $x \in X$ and $n \in \mathbb{N}$.*

We require the following elementary properties of local spectral subspace.

Lemma 1.2. [1] *Let $T \in B(X)$ and $F \subseteq \mathbb{C}$, then $X_T(F)$ is a T -hyperinvariant subspace of X , and*

$$(T - \lambda I)X_T(F) = X_T(F) \quad \forall \lambda \in \mathbb{C} \setminus F.$$

The following Lemma is a key of the proofs coming after.

Lemma 1.3. [2] *Let x be a nonzero vector in X and $T, S \in B(X)$. If $X_T(\{\lambda\}) = X_S(\{\lambda\})$ for all $\lambda \in \mathbb{C}$. Then, $\sigma_T(x) = \{\mu\}$ if and only if $\sigma_S(x) = \{\mu\}$ for all $\mu \in \mathbb{C}$.*

In this paper, we describe surjective linear maps φ on $B(X)$ that satisfy

$$X_{\varphi(T)}(\{0\}) = X_T(\{0\}), \quad \forall T \in Q(X).$$

Furthermore, we characterize maps φ (not necessarily linear nor surjective) on $B(X)$ which satisfy $X_{\varphi(T)-\varphi(S)}(\{0\}) = X_{T-S}(\{0\})$ for every $T, S \in B(X)$ which $T - S \in Q(X)$.

2. MAIN RESULTS

The following lemma gives an explicit identification of local spectral subspace in the case of rank-one operator.

Lemma 2.1. [4] *Let $R \in F_1(X)$ be a non-nilpotent operator, and let λ be a nonzero eigenvalue of R . Then $X_R(\{0\}) = \ker(R)$ and $X_R(\{\lambda\}) = \text{Im}(R)$.*

The following Lemma is a key of the proofs next theorem.

Lemma 2.2. *Let $\varphi : B(X) \longrightarrow B(X)$ be a surjective linear map. If satisfies one of the following assertion:*

$$(a) X_{\varphi(T)}(\{0\}) \subseteq X_T(\{0\}), \quad \forall T \in Q(X)$$

$$(b) X_T(\{0\}) \subseteq X_{\varphi(T)}(\{0\}), \quad \forall T \in Q(X)$$

then there exists a nonzero scalar $\lambda \in \mathbb{C}$ such that $\varphi(I) = \lambda I$, where I stands for the identity operator on X .

Theorem 2.3. *Let $\varphi : B(X) \longrightarrow B(X)$ be a surjective linear map. Then the following assertions are equivalent.*

$$(a) X_{\varphi(T)}(\{0\}) \subseteq X_T(\{0\}), \quad \forall T \in Q(X)$$

$$(b) X_T(\{0\}) \subseteq X_{\varphi(T)}(\{0\}), \quad \forall T \in Q(X)$$

(c) there exists a nonzero scalar $\lambda \in \mathbb{C}$ such that $\varphi(T) = \lambda T$, for all $T \in Q(X)$.

In the following lemma and theorem we give a concrete form of maps that preserve the local subspace of the difference of two operators associated with $\{0\}$.

Lemma 2.4. *Let $\varphi : B(X) \longrightarrow B(X)$ be a map which satisfies $X_{\varphi(T)-\varphi(S)}(\{0\}) = X_{T-S}(\{0\})$ for every $T, S \in B(X)$ which $T - S \in Q(X)$. Then for every nonzero scalar $\lambda \in \mathbb{C}$, there exists a nonzero scalar $\mu_\lambda \in \mathbb{C}$ such that $\varphi(\lambda I) = \mu_\lambda I + \varphi(0)$.*

Theorem 2.5. *Let $\varphi : B(X) \longrightarrow B(X)$ be a map. Then the following assertions are equivalent.*

$$(a) X_{\varphi(T)-\varphi(S)}(\{0\}) = X_{T-S}(\{0\}) \text{ for every } T, S \in B(X) \text{ which } T - S \in Q(X)$$

(b) there exists a nonzero scalar $\lambda \in \mathbb{C}$ such that $\varphi(T) = \lambda T + \varphi(0)$, for all $T \in Q(X)$.

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