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WELL-POSEDNESS OF THE ZKB EQUATION IN THE WEIGHTED SPACES

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Abstract. In this work we consider a generalized dissipative ZK equation. The associated linear part produces both semigroup and group. As the dissipation is directional, we use a regularization method to study the associated initial value problem in Sobolev spaces $H^s(\mathbb{R}^n)$ and some weighted spaces $\mathcal{F}_r^{s,p}$. We also prove an ill-posedness result in the two-dimensional case.

1. Introduction

In this paper, we study of the following evolution equation

$$u_t + (L_\alpha + f(u))_x = 0, \qquad t \in \mathbb{R}^+, \tag{1.1}$$

where $L_\alpha = \Delta u - \alpha u_x$ is the ZK operator with a directional dissipation. Here, $\alpha \in \mathbb{R}$, f is a differentiable real-valued function on $\mathbb{R}$ such that $f(0) = 0$ and $f'(0) = 0$ and we consider $u = u(x, y, t)$ such that $(x, y) \in \mathbb{R} \times \mathbb{R}^{n-1}$, $n \geq 2$. We also assume that $f(x) = O(x^{p+1})$, for $p \in \mathbb{N}$. The evolution equation (1.1) is known as the ZKB equation when $f(u) = u^2/2$, because of appearing the ZK operator and the Burgers-type dissipation. The ZKB equation (1.1) describes asymptotically the propagations of nonlinear dust acoustic waves in a nonuniform

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magnetized dusty plasma [3, 1]. By neglecting the dissipative term in (1.1), we will get the so-called ZK equation.

$$u_t + (\Delta u + f(u))_x = 0. \tag{1.2}$$

In this work, we are going to study the Cauchy problem associated to (1.1) in Sobolev spaces. Our strategy is to use a regularization by applying more dissipative terms to the equation; in fact, we will consider the following regularized ZKB (rZKB) problem:

$$u_t + (\Delta u + f(u) - \alpha u_x)_x - \beta \Delta_\perp u = 0, \tag{1.3}$$

where $\beta \in \mathbb{R}^+$ and $\triangle = \partial_x^2 + \triangle_\perp$. Next, by using the semigroup properties of (1.3), we endeavor to prove a well-posedness result in $H^s(\mathbb{R}^2)$ spaces for $s > 2$ and show our results hold in weak topology as the parameter β tends to zero. In dimension two, we will also show that (1.1) is well-posed in the weighted spaces $H^s(\mathcal{W})$ (see Definition 2.1) for some suitable weight functions $\mathcal{W}$. Regarding on the ill-posedness issue, we are not able to derive a criterion to anticipate the minimum index of local well-posedness due to the directional dissipation; but we establish that the flow-map of equation (1.1) fails to be C^2 in $H^{s,0}(\mathbb{R}^2)$ for $s < -\frac{3}{4}$.

2. Main results

Now we summarize our main results skipping several propositions and technical lemmas.

Definition 2.1. We denote by $L^2(\mathcal{W})$ the space of all real-valued functions f such that $\|f\|^2_{L^2(\mathcal{W}dx)} = \int f^2(x)\mathcal{W}(x)\ dx < \infty$, where $H^s = H^s(\mathbb{R}^n)$ is the nonhomogeneous Sobolev space. Especially, for $\mathcal{W}(x) = 1 + \sum_{i=1}^{n} x_i^{2r_i}$ and $r = (r_1, \cdots, r_n) \in \mathbb{R}^n$, we denote $\mathcal{F}_r^s$ the space of all real-valued measurable functions f such that $\|f\|_{\mathcal{F}_r^s} = \|f\|_{H^s} + \|f\|_{L^2(\mathcal{W})} < \infty$. Similarly for any $p \geq 1$, one can define $\mathcal{F}_r^{s,p} = H^s \cap L^p(\mathcal{W})$. For $r \in \mathbb{R}$, we denote $\mathcal{F}_r^s$ as $\mathcal{F}_{r,\cdots,r}^s$ and $H^s(\mathcal{W}) = H^s \cap L^2(\mathcal{W})$.

Using properties of the semigroup associated to the linear problem, we can obtain the our main local well-posedness theorem.

Theorem 2.2. *Let $s > 2$. Then for any initial data $u_0 \in H^s$, there exist $T^s_{\alpha,\beta} = T\left(\alpha, \beta, \|u_0\|_{H^s}\right)$ and a unique solution of the initial value problem (1.3), $u_{\alpha,\beta}(\cdot)$, defined in the interval $\left[0, T^s_{\alpha,\beta}\right]$ satisfying $u_{\alpha,\beta} \in C\left(\left[0, T^s_{\alpha,\beta}\right]; H^s\right) \cap C^1\left(\left[0, T^s_{\alpha,\beta}\right]; H^{s-2}\right)$. Moreover,*

$$u_{\alpha,\beta} \in C\left(\left(0, T^s_{\alpha,\beta}\right]; H^\infty\right).$$

Furthermore, the theorem is true for $\beta = 0$ (in the weak topology sense) and $\alpha = \beta = 0$.

To prove Theorem 2.2, we use the following estimates. The semigroup associated with (1.1) is denoted by $U_{\alpha,\beta}$.

Lemma 2.3. *Let $\alpha, \beta > 0$ and $s \in \mathbb{R}$, then for any $\delta \geq 0$ and all $t > 0$, $U_{\alpha,\beta}(t) \in \mathcal{L}(H^s, H^{s+\delta})$. Moreover there exists $C_s > 0$ such that*

$$\|u_{\alpha,\beta}(t)\|_{H^{s+\delta}} \leq C_s \sqrt{1 + t^{-s} \max\{\alpha^{-s}, \beta^{-s}\}}\, \|u_0\|_{H^s}, \tag{2.1}$$

for any $u_0 \in H^s$.

Lemma 2.4. *Let $U^0_{\alpha,\beta}(t) = U_{\alpha,\beta}(t)\delta_0$, $m = (m_1, m_2), k = (k_1, k_2) \in (\mathbb{Z}^+)^2$, $x \in \mathbb{R}^2$ and $t > 0$, where δ_0 is Dirac delta.*

i) *If $2 \leq p \leq \infty$, then there exists $C(\alpha, \beta) > 0$ such that*

$$\left\|x^k D^m U^0_{\alpha,\beta}(t)\right\|_{L^p} \leq C(\alpha, \beta)\langle t\rangle^{\frac{1}{2}|k|}\, t^{-\frac{1}{2}|m| - 2\left(1 - \frac{1}{p}\right)}. \tag{2.2}$$

ii) *If $1 \leq p \leq 2$, then there exists $C(\alpha, \beta) > 0$ such that*

$$\left\|x^k D^m U^0_{\alpha,\beta}(t)\right\|_{L^p} \leq C(\alpha, \beta)\langle t\rangle^{\frac{1}{2}(|k|-1)}\, t^{-2\left(1 - \frac{1}{p}\right) - \frac{|m|}{2}}, \tag{2.3}$$

where $|k| = k_1 + k_2$, $|m| = m_1 + m_2$ and $\langle t\rangle = (1 + t^2)^{1/2}$.

iii) *$u_{\alpha,\beta}(t) \in L^p$ for any $2 \leq p \leq \infty$, if $u_0 \in L^2$. Moreover,*

$$\|u_{\alpha,\beta}(t)\|_{L^p} \lesssim t^{-\theta}\|u_0\|_{L^p},$$

where $\theta = \theta(p) = 1 - \frac{2}{p}$.

Now, we use the properties of the Kato-Ponce commutator [2]. Let J^s be is the Bessel potential of order $-s$ and $\mathcal{S}(\mathbb{R}^2)$ is the Schwartz class.

Lemma 2.5. *If $f, g \in \mathcal{S}(\mathbb{R}^2)$, $s > 0$ and $p \in (1, +\infty)$, then*

$$\|[J^s, M_f]g\|_{L^p} \lesssim \left(\|\nabla f\|_{L^{p_1}} \left\|J^{s-1}g\right\|_{L^{p_2}} + \|J^s f\|_{L^{p_3}} \|g\|_{L^{p_4}}\right), \tag{2.4}$$

$$\|fg\|_{L^p} \lesssim \left(\|f\|_{L^{p_1}} \|J^s g\|_{L^{p_2}} + \|J^s f\|_{L^{p_3}} \|g\|_{L^{p_4}}\right), \tag{2.5}$$

where $p_2, p_3 \in (1, +\infty)$ such that

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

Theorem 2.6 (Continuous Dependence)**.** *There exists a metric space $E^s_{\alpha,\beta}$ such that for $R > 0$, the correspondence $u_0 \to u_{\alpha,\beta}$ that associates to $u_0 \in \mathcal{B}_R$ the solution $u_{\alpha,\beta}$ of (1.3) with initial data u_0 is continuous mapping of $\mathcal{B}_R$ to $E^s_{\alpha,\beta}$, where $\mathcal{B}_R$ is the ball of radius R centered at the origin of H^s.*

To study the well-posedness in the weighted spaces, we need to understand the behavior of our semigroup in such spaces.

Remark 2.7. One can obtain the explicit form of $\mathcal{X}(t)$, T^s and $\mathcal{A}_T$. Indeed,

$$\mathcal{X}(t) = \frac{2^{\frac{2}{p}}\|u_0\|_{H^s}^2}{(2 - c_s tp\|u_0\|_{H^s}^p)^{\frac{2}{p}}},$$

$$T^s = \frac{2}{(c_s p\|u_0\|_{H^s}^2)} \text{ and } \mathcal{A}_T = \frac{2^{\frac{1}{p}}\|u_0\|_{H^s}}{(2 - c_s pT\|u_0\|_{H^s}^p)^{\frac{1}{p}}}, \text{ for any } T \in (0, T^s).$$

Lemma 2.8. *Let $p, m \in \mathbb{N}$, $\beta > 0$, $t > 0$ and $\omega = (\omega_1, \omega_2) \in \mathbb{R}^2$. Then there exists $C(m, \beta, |\omega|) > 0$ such that for any $f \in \mathcal{F}_{0,m}^{0,p}$,*

$$\|D^\omega U_{\alpha,\beta}(t) f\|_{\mathcal{F}_{0,m}^{0,p}} \le C(m, \beta, |\omega|)\left(1 + t^{-\frac{|\omega|}{2}} + t^{\frac{m-|\omega|}{2}}\right)\|f\|_{\mathcal{F}_{0,m}^{0,p}}$$

Theorem 2.9 (Well-posedness Result in Weighted Spaces)**.** *Let $\mathcal{W}$ be a weight with all its first and second derivatives bounded and such that $|\mathcal{W}(x,y)| \le C_\varepsilon e^{\varepsilon\left(x^2+y^2\right)}$, for all $(x,y) \in \mathbb{R}^2$ and any $\varepsilon \in (0, \widetilde{\varepsilon})$, for some $\widetilde{\varepsilon} > 0$ and $C_\varepsilon > 0$. Let also $u_0 \in H^s(\mathcal{W})$, $s > 2$. Then the solution $u_{\alpha,\beta}$ of the equation (1.3) corresponding to the initial data u_0 is in $C\left(\left[0, T_{\alpha,\beta}^s\right); H^s(\mathcal{W})\right)$. Moreover, the continuous dependence of solutions of the equation (1.1) holds in $H^s(\mathcal{W})$.*

Theorem 2.10 (Persistence of Solutions)**.** *Let $s \in \mathbb{N}$, $s \ge 3$ and $\beta \ge 0$. Also suppose that $u_{\alpha,\beta} \in C\left(\left[0, T_{\alpha,\beta}^s\right); H^s\right)$ is the maximal solution of the rZKB equation corresponding to the initial data $u_0 \in \mathcal{F}_{1,s}^{s,2}$. Then $u_{\alpha,\beta} \in C\left(\left[0, T_{\alpha,\beta}^s\right); \mathcal{F}_{1,s}^{s,2}\right)$.*

Next, we show that the Picard iteration method cannot be used to obtain a solution of (1.1). Indeed, we construct a sequence of initial data that will ensure the irregularity of the flow map for $s < -3/4$.

Theorem 2.11 (Ill-posedness Result)**.** *Let $s < -\frac{3}{4}$ and $H^{s,0}(\mathbb{R}^2)$ be the x-directional Sobolev space. Then there is no $T > 0$ such that the ZKB equation (1.1), with $f(u) = u^2/2$, admits a unique solution u in $C([0,T]; H^{s,0}(\mathbb{R}^2))$ for any initial data in the same ball of $H^{s,0}(\mathbb{R}^2)$ centered at the origin and the map $\phi \to u$ is C^2-differentiable at the origin from $H^{s,0}(\mathbb{R}^2)$ to $C([0,T]; H^{s,0}(\mathbb{R}^2))$.*

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