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CLOSED RANGE WEIGHTED COMPOSITION OPERATORS ON THE HARDY AND WEIGHTED BERGMAN SPACES

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Abstract. In this paper, we investigate bounded below weighted composition operators $C_{\psi,\varphi}$ on a Hilbert space of analytic functions. Then for $\psi \in H^\infty$ and a univalent map φ, we characterize all closed range weighted composition operators $C_{\psi,\varphi}$ on H^2 and A^2_α. Also we show that for $\psi \in H^\infty$ which is bounded away from zero near the unit circle, the weighted composition operator $C_{\psi,\varphi}$ is bounded below on H^2 or A^2_α if and only if C_φ has closed range.

1. Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane. The Hardy space H^2 is the set of all analytic functions f on $\mathbb{D}$ such that

$$\|f\|_1^2 = \lim_{r\to 1}\int_0^{2\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} < \infty.$$

We recall that $H^\infty(\mathbb{D}) = H^\infty$ is the space of all bounded analytic functions defined on $\mathbb{D}$, with supremum norm $\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|$.

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Let dA be the normalized area measure in $\mathbb{D}$. The weighted Bergman spaces $A^2_\alpha(\mathbb{D}) = A^2_\alpha$, for $\alpha > -1$, are defined by

$$A^2_\alpha(\mathbb{D}) = \{f \text{ analytic in } \mathbb{D} : ||f||^2_{\alpha+2} = \int_{\mathbb{D}} |f|^2 dA_\alpha < \infty\},$$

where $dA_\alpha = (\alpha+1)(1-|z|^2)^\alpha dA$. We know that for $\alpha > -1$, A^2_α is a Hilbert space. The case when $\alpha = 0$ is known as the (unweighted) Bergman space, and is often denoted simply A^2.

Let φ be an analytic map from the open unit disk $\mathbb{D}$ into itself. The operator that takes the analytic map f to $f \circ \varphi$ is a composition operator and is denoted by C_φ. A natural generalization of a composition operator is an operator that takes f to $\psi \cdot f \circ \varphi$, where ψ is a fixed analytic map on $\mathbb{D}$. This operator is aptly named a weighted composition operator and is usually denoted by $C_{\psi,\varphi}$. More precisely, if z is in the unit disk then $(C_{\psi,\varphi} f)(z) = \psi(z) f(\varphi(z))$. In this paper, we assume that $C_{\psi,\varphi}$ is a bounded operator.

The automorphisms of $\mathbb{D}$, that is, the one-to-one analytic maps of the disk onto itself, are just the functions $\varphi(z) = \lambda \frac{a-z}{1-\bar{a}z}$, where $|\lambda| = 1$ and $|a| < 1$. We denote the class of automorphisms of $\mathbb{D}$ by $\mathrm{Aut}(\mathbb{D})$.

Closed range composition operators were studied on the Hardy and weighted Bergman spaces in [1], [2], [4] and [5]. In the second section, we investigate closed range weighted composition operators. We show that if $C_{\psi,\varphi}$ is bounded below on a Hilbert space of analytic functions, then C_φ has closed range. Next, we show that $C_{\psi,\varphi}$ is bounded below on H^2 or A^2_α if and only if C_φ has closed range, when $\psi \in H^\infty$ is bounded away from zero near the unit circle. In Theorem 2.5, for $\psi \in H^\infty$ and φ which is a univalent, holomorphic self-map of $\mathbb{D}$, we determine all closed range operators $C_{\psi,\varphi}$ on H^2 and A^2_α. In this paper, we state some results of [3].

2. Main results

Let H be a Hilbert space. The set of all bounded operators from H into itself is denoted by $B(H)$. We say that an operator $A \in B(H)$ is bounded below if there is a constant $c > 0$ such that $c\|h\| \leq \|A(h)\|$ for all $h \in H$.

In the next lemma, we show that for operators A and B which have closed range, if A is bounded below, then AB has closed range.

Lemma 2.1. *Suppose that A and B belong to $B(H)$. Let A and B have closed range. If A is bounded below, then AB has closed range.*

If f is defined on a set V and if there is a positive constant m so that $|f(z)| \geq m$, for all z in V, we say f is bounded away from zero on V. In particular, we say that ψ is bounded away from zero near the unit circle, that is, there are $\delta > 0$ and $\epsilon > 0$ such that

$$|\psi(z)| > \epsilon \text{ for } \delta < |z| < 1.$$

In Propositions 2.2 and 2.3 and Theorem 2.5, we investigate bounded below weighted composition operators

Proposition 2.2. *Suppose that H is a Hilbert space of analytic functions. If $C_{\psi,\varphi}$ is bounded below on H, then C_φ has closed range.*

Suppose that φ is a constant function. Then the range of φ on $\mathbb{D}$ misses a neighborhood of the unit circle. [5, Corollary 4.2] and [5, Example 1] imply that C_φ does not have a closed range on H^2 and A^2_α. We use this fact in Theorem 2.3.

Theorem 2.3. *Let $\psi \in H^\infty$ be bounded away from zero near the unit circle. The composition operator C_φ has closed range if and only if $C_{\psi,\varphi}$ is bounded below on H^2 or A^2_α.*

In [2, Theorem 5.1], Akeroyd et al. found the following proposition. We use it in the next theorem in order to characterize all closed range weighted composition operators $C_{\psi,\varphi}$ on H^2 and A^2_α, when φ is univalent.

Proposition 2.4. *LSuppose that φ is a univalent, holomorphic self-map of $\mathbb{D}$. Then C_φ has closed range on H^2 or A^2_α if and only if φ is an automorphism of $\mathbb{D}$.*

If $\psi \equiv 0$, then it is easy to see that $C_{\psi,\varphi}$ has closed range. Therefore, in the following theorem, we assume that $\psi \not\equiv 0$.

Theorem 2.5. *Assume that φ is a univalent, holomorphic self-map of $\mathbb{D}$. Let $\psi \in H^\infty$ and $\psi \not\equiv 0$. The weighted composition operator $C_{\psi,\varphi}$ has closed range on H^2 if and only if φ is an automorphism of $\mathbb{D}$ and there exits a constant $m > 0$ such that $|\psi| \geq m$ almost everywhere on $\partial\mathbb{D}$. Moreover, the weighted composition operator $C_{\psi,\varphi}$ has closed range on A^2_α if and only if φ is an automorphism of $\mathbb{D}$ and $\psi = hb$, where $h \in H^\infty$ is invertible in H^∞ and b is a finite product of interpolating Blaschke products.*

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