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WHEN THE ADJOINT OF A WEIGHTED COMPOSITION OPERATOR IS BOUNDED BELOW

M. HAJI SHAABANI

Department of Mathematics, Shiraz University of Technology, P. O. Box 71555-313, Shiraz, Iran.
shaabani@sutech.ac.ir

ABSTRACT. In this paper, we obtain that $C^*_{\psi,\varphi}$ is bounded below on H^2 or A^2_α if and only if $C_{\psi,\varphi}$ is invertible.

1. INTRODUCTION

Let $\mathbb{D}$ denote the open unit disk in the complex plane. For $\alpha > -1$, the weighted Bergman space $A^2_\alpha(\mathbb{D}) = A^2_\alpha$ is the set of functions f analytic in $\mathbb{D}$ with

$$\|f\|^2_{\alpha+2} = (\alpha+1)\int_{\mathbb{D}} |f(z)|^2(1-|z|^2)^\alpha dA(z) < \infty,$$

where dA is the normalized area measure in $\mathbb{D}$. The case when $\alpha = 0$ is known as the (unweighted) Bergman space, and is often denoted simply A^2.

The Hardy space, denoted $H^2(\mathbb{D}) = H^2$, is the set of all analytic functions f on $\mathbb{D}$, satisfying the norm condition

$$\|f\|^2_1 = \lim_{r\to 1}\int_0^{2\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} < \infty.$$

The space $H^\infty(\mathbb{D}) = H^\infty$ consists of all the functions that are analytic and bounded on $\mathbb{D}$, with supremum norm $\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|$.

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Let φ be an analytic map from the open unit disk $\mathbb{D}$ into itself. The operator that takes the analytic map f to $f \circ \varphi$ is a composition operator and is denoted by C_φ. A natural generalization of a composition operator is an operator that takes f to $\psi \cdot f \circ \varphi$, where ψ is a fixed analytic map on $\mathbb{D}$. This operator is aptly named a weighted composition operator and is usually denoted by $C_{\psi,\varphi}$. More precisely, if z is in the unit disk then $(C_{\psi,\varphi}f)(z) = \psi(z)f(\varphi(z))$.

Suppose that H and H' are Hilbert spaces and $A : H \to H'$ is a bounded operator. The operator A is said to be left semi-Fredholm if there is a bounded operator $B : H' \to H$ and a compact operator K on H such that $BA = I + K$. Analogously, A is right semi-Fredholm if there is a bounded operator $B' : H' \to H$ and a compact operator K' on H' such that $AB' = I + K'$. An operator A is said to be Fredholm if it is both left and right semi-Fredholm. It is not hard to see that A is left semi-Fredholm if and only if A^* is right semi-Fredholm. Hence A is Fredholm if and only if A^* is Fredholm. Note that an invertible operator is Fredholm. By using the definition of Fredholm operator, it is not hard to see that if the operators A and B are Fredholm on a Hilbert space H, then AB is also Fredholm on H.

The automorphisms of $\mathbb{D}$, that is, the one-to-one analytic maps of the disk onto itself, are just the functions $\varphi(z) = \lambda\frac{a-z}{1-\bar{a}z}$, where $|\lambda| = 1$ and $|a| < 1$. We denote the class of automorphisms of $\mathbb{D}$ by $\mathrm{Aut}(\mathbb{D})$. Automorphisms of $\mathbb{D}$ take $\partial\mathbb{D}$ onto $\partial\mathbb{D}$. It is known that C_φ is Fredholm on the Hardy space if and only if $\varphi \in \mathrm{Aut}(\mathbb{D})$ (see [1]).

In the second section, we investigate Fredholm and invertible weighted composition operators. In Theorem 2.7, we show that the operator $C^*_{\psi,\varphi}$ is bounded below on H^2 or A^2_α if and only if $C_{\psi,\varphi}$ is invertible. In this paper, we state some results of [4].

2. Main results

Let H be a Hilbert space. The set of all bounded operators from H into itself is denoted by $B(H)$. We say that an operator $A \in B(H)$ is bounded below if there is a constant $c > 0$ such that $c\|h\| \leq \|A(h)\|$ for all $h \in H$.

If f is defined on a set V and if there is a positive constant m so that $|f(z)| \geq m$, for all z in V, we say f is bounded away from zero on V. In particular, we say that ψ is bounded away from zero near the unit circle, that is, there are $\delta > 0$ and $\epsilon > 0$ such that

$$|\psi(z)| > \epsilon \text{ for } \delta < |z| < 1.$$

Now we state the following simple and well-known lemma, and we frequently use it in this paper.

Lemma 2.1. *Let $C_{\psi,\varphi}$ be a bounded operator on H^2 or A^2_α. Then, for each $w \in \mathbb{D}$, $C^*_{\psi,\varphi} K_w = \overline{\psi(w)} K_{\varphi(w)}$.*

Lemma 2.2. *Suppose that A and B are two bounded operators on a Hilbert space H. If AB is a Fredholm operator, then B is left semi-Fredholm.*

Zhao in [5] characterized Fredholm weighted composition operators on H^2. Also Zhao in [6] found necessary conditions of φ and ψ for a weighted composition operator $C_{\psi,\varphi}$ on A^2_α to be Fredholm. In the following proposition, we obtain a necessary and sufficient condition for $C_{\psi,\varphi}$ to be Fredholm on H^2 and A^2_α. The idea of the proof of the next proposition is different from [5] and [6].

Proposition 2.3. *The operator $C^*_{\psi,\varphi}$ is left semi-Fredholm on H^2 or A^2_α if and only if $\varphi \in Aut(\mathbb{D})$ and $\psi \in H^\infty$ is bounded away from zero near the unit circle. Under this conditions $C_{\psi,\varphi}$ is a Fredholm operator.*

In the next proposition, we find a necessary condition of ψ for an operator $C^*_{\psi,\varphi}$ to be bounded below on H^2 and A^2_α. Then we use Proposition 2.4 in order to obtain all invertible weighted composition operators on H^2 and A^2_α.

Proposition 2.4. *Let ψ be an analytic map of $\mathbb{D}$ and φ be an analytic self-map of $\mathbb{D}$. If $C^*_{\psi,\varphi}$ is bounded below on H^2 or A^2_α, then $\psi \in H^\infty$ is bounded away from zero on $\mathbb{D}$ and $\varphi \in Aut(\mathbb{D})$.*

Bourdon in [2, Theorem 3.4] obtained the following corollary; we give another proof (see also [3, Theorem 2.0.1]).

Corollary 2.5. *Let ψ be an analytic map of $\mathbb{D}$ and φ be an analytic self-map of $\mathbb{D}$. The weighted composition operator $C_{\psi,\varphi}$ is invertible on H^2 or A^2_α if and only if $\varphi \in Aut(\mathbb{D})$ and $\psi \in H^\infty$ is bounded away from zero on $\mathbb{D}$.*

Note that if $C_{\psi,\varphi}$ is invertible, then $C^*_{\psi,\varphi}$ is bounded below. Hence by Proposition 2.4 and Corollary 2.5, we can see that $C^*_{\psi,\varphi}$ is bounded below if and only if $C_{\psi,\varphi}$ is invertible.

The algebra $A(\mathbb{D})$ consists of all continuous functions on the closure of $\mathbb{D}$ that are analytic on $\mathbb{D}$. In the next corollary, we find some Fredholm weighted composition operators which are not invertible.

Corollary 2.6. *Suppose that $\varphi \in Aut(\mathbb{D})$ and $\psi \in A(\mathbb{D})$. Assume that $\{z \in \mathbb{D} : \psi(z) = 0\}$ is a nonempty finite set and for each $z \in \partial\mathbb{D}$, $\psi(z) \neq 0$. Then $C_{\psi,\varphi}$ is Fredholm, but it is not invertible.*

Theorem 2.7. *Suppose that ψ is an analytic map of $\mathbb{D}$ and φ is an analytic self-map of $\mathbb{D}$. The operator $C^*_{\psi,\varphi}$ is bounded below on H^2 or A^2_α if and only if $\varphi \in Aut(\mathbb{D})$ and $\psi \in H^\infty$ is bounded away from zero on $\mathbb{D}$.*

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