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NORMS OF COMPOSITION–DIFFERENTIATION OPERATORS ON THE HARDY SPACE

M. FATEHI AND C. N. B. HAMMOND

Department of Mathematics
Shiraz Branch, Islamic Azad University
Shiraz, Iran
fatehimahsa@yahoo.com
Department of Mathematics and Statistics
Connecticut College
New London, CT 06320
cnham@conncoll.edu

Abstract. Let φ be a nonconstant analytic self-map of the open unit disk in $\mathbb{C}$, with $\|\varphi\|_\infty < 1$. Consider the operator D_φ, acting on the Hardy space H^2, given by differentiation followed by composition with φ. We obtain results relating to the norm of such an operator.

1. Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane. The *Hardy space* H^2 is the Hilbert space consisting of all analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on $\mathbb{D}$ such that

$$\|f\| = \sum_{n=0}^{\infty} |a_n|^2 < \infty.$$

We write H^∞ to denote the space of all bounded analytic functions on $\mathbb{D}$, with $\|f\|_\infty = \sup\{|f(z)| : z \in \mathbb{D}\}$.

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For an analytic map $\varphi\colon \mathbb{D} \to \mathbb{D}$, the *composition operator* C_φ is defined by the rule $C_\varphi(f) = f \circ \varphi$. Every composition operator is bounded on H^2, with

$$\sqrt{\frac{1}{1-|\varphi(0)|^2}} \leq \|C_\varphi\| \leq \sqrt{\frac{1+|\varphi(0)|}{1-|\varphi(0)|}}.$$

(See, for example, [2, Corollary 3.7].) For a function ψ in H^∞, the *Toeplitz operator* T_ψ is defined $T_\psi(f) = \psi \cdot f$. Every such operator is bounded on H^2, with $\|T_\psi\| = \|\psi\|_\infty$ (see [7, Theorem 5]).

In the context of analytic functions on $\mathbb{D}$, it also seems reasonable to consider operators defined in terms of differentiation. It is easy to see that the differentiation operator $D(f) = f'$ is unbounded on the Hardy space: $\|D(z^n)\|/\|z^n\| = n$ for any natural number n. Nevertheless, for many analytic maps $\varphi\colon \mathbb{D} \to \mathbb{D}$, the operator

$$f(z) \mapsto f'(\varphi(z)) \tag{1.1}$$

is bounded on H^2. Many authors, following the example of [5] and [6], have used the notation $C_\varphi D$ to denote such an operator. Because of the unboundedness of D, it makes sense to write (1.1) as a single operator, particularly when that operator is bounded on H^2. We will write D_φ to denote the operator on H^2 given by the rule

$$D_\varphi(f) = f' \circ \varphi.$$

We will refer to such an operator as a *composition–differentiation operator*. The Closed Graph Theorem shows that D_φ is bounded on H^2 whenever D_φ takes H^2 into itself.

Ohno [6] established a basic set of results relating to when the operators we are calling D_φ are bounded or compact on H^2. We will only be considering φ with $\|\varphi\|_\infty < 1$, in which case D_φ is guaranteed to be Hilbert–Schmidt on H^2, and hence both bounded and compact (see [6, Theorem 3.3]). There are instances of bounded or compact D_φ with $\|\varphi\|_\infty = 1$, but they are beyond the scope of our current investigation.

The purpose of this note is to explore the operators D_φ in more detail. In particular, we find a representation for the adjoint D_φ^* when φ is linear fractional (Theorem 2.1). In the specific case where $\rho(z) = rz$ for $0 < |r| < 1$, we compute the norm $\|D_\rho\|$ explicitly (Theorem 2.2). Applying established results relating to composition operators, we also obtain estimates for the norm of D_φ whenever $\|\varphi\|_\infty < 1$ (Theorem 2.3). In this paper, we state some results of [3]. Moreover, normality and self-adjointness of a slightly broader class of composition–differentiation operators were investigated in [4].

For any point w in $\mathbb{D}$, define $K_w(z) = \frac{1}{1-\overline{w}z}$. It is well known that K_w acts as the reproducing kernel function for point-evaluation:

$$\langle f, K_w\rangle = f(w)$$

for any f in H^2. In a similar manner, define

$$K_w^{(1)}(z) = \frac{z}{(1-\overline{w}z)^2}.$$

Observe that $K_w^{(1)}$ acts as the reproducing kernel for point-evaluation of the first derivative:

$$\left\langle f, K_w^{(1)}\right\rangle = f'(w)$$

2. Main results

The goal of this section is to obtain information about the adjoint and norm of D_φ in certain specific instances. If $\varphi(z) = \frac{az+b}{cz+d}$ is a non-constant linear fractional self-map of $\mathbb{D}$, then the map $\sigma(z) = \frac{\overline{a}z-\overline{c}}{-\overline{b}z+\overline{d}}$. also takes $\mathbb{D}$ into itself (see [1, Lemma 1]). It is not difficult to show that $\|\sigma\|_\infty < 1$ whenever $\|\varphi\|_\infty < 1$. The relationship between these two maps has long been considered in reference to the adjoints of composition operators. In the context of composition–differentiation operators, we obtain the following formula.

Theorem 2.1. *For a pair of linear fractional maps φ and σ, as described above, $D_\varphi^* T^*_{K^{(1)}_{\sigma(0)}} = T_{K^{(1)}_{\varphi(0)}} D_\sigma$.*

This result bears a close resemblance to Cowen's adjoint formula for composition operators (see [1, Theorem 2]), which can be rewritten $C_\varphi^* T^*_{K_{\sigma(0)}} = T_{K_{\varphi(0)}} C_\sigma$.

Theorem 2.2. *If $\rho(z) = rz$ for some real number $0 < r < 1$, then*

$$\|D_\rho\| = \left\lfloor \frac{1}{1-r} \right\rfloor r^{\lfloor 1/(1-r)\rfloor - 1}, \tag{2.1}$$

where $\lfloor \cdot \rfloor$ denotes the greatest integer function.

There are several interesting consequences of Theorem 2.2. First of all, $\|D_\rho\| = 1$ for $0 < r \leq 1/2$ and $\|D_\rho\| > 1$ for $1/2 < r < 1$. Secondly, $\|D_\rho\|$ tends to ∞ as r goes to 1. Since composition with a rotation is an isometry, (2.1) holds with r replaced by $|r|$ for any complex number r with $0 < |r| < 1$. Likewise, the same formula holds for $\|D_\varphi\|$ where $\varphi(z) = rz^k$ for any k in $\mathbb{N}$.

Let $\|\varphi\|_\infty \le r < 1$ and define $\varphi_r = (1/r)\varphi$. Observe that

$$D_\varphi = C_{\varphi_r} D_\rho. \tag{2.2}$$

Since $\|D_\varphi\| \le \|C_{\varphi_r}\|\|D_\rho\|$, we obtain the following estimate for $\|D_\varphi\|$.

Theorem 2.3. *If φ is a nonconstant analytic self-map of $\mathbb{D}$, with $\|\varphi\|_\infty < 1$, then*

$$\sqrt{\frac{1+|\varphi(0)|^2}{\left(1-|\varphi(0)|^2\right)^3}} \le \|D_\varphi\| \le \sqrt{\frac{r+|\varphi(0)|}{r-|\varphi(0)|}} \left\lfloor \frac{1}{1-r} \right\rfloor r^{\lfloor 1/(1-r)\rfloor - 1},$$

whenever $\|\varphi\|_\infty \le r < 1$.

Example 2.4. If $\|\varphi\|_\infty \le 1/2$, we may take $r = 1/2$ to see that

$$\sqrt{\frac{1+|\varphi(0)|^2}{\left(1-|\varphi(0)|^2\right)^3}} \le \|D_\varphi\| \le \sqrt{\frac{1+2|\varphi(0)|}{1-2|\varphi(0)|}}.$$

In particular, $\|D_\varphi\| = 1$ whenever both $\|\varphi\|_\infty \le 1/2$ and $\varphi(0) = 0$.

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