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THE BESOV SPACES ASSOCIATED WITH OPERATORS ON ANALYTIC FUNCTIONS

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Abstract. In this paper we aim to investigate connections between operators on analytic functions with Besov spaces defined on unit disc and study their basic properties with attention to the maclurian coefficients.

1. Introduction

We denote by $\mathbb{D}$ the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{H}(\mathbb{D})$ the class of all analytic functions on $\mathbb{D}$. Also let $\mathcal{A}$ be the class of all elments of $\mathcal{H}(\mathbb{D})$ such as f in which $f(0) = 0 = f'(0) - 1$.

We denote by $\mathbb{B}$ (Bloch space) the subclass of $\mathcal{H}(\mathbb{D})$ consists of all analytic functions f such that

$$\|f\|_B := |f(0)| + \sup_{z\in\mathbb{D}}(1 - |z|^2)|f'(z)| < \infty.$$

Many of operators in Fractional calculus have application in the other branches of Math. Also there are many results due to operators in the Fractional calculus.We motivated by [3] define the Fractional derivative of order λ, by

$$D_z^\lambda f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\omega)}{(z-\omega)^\lambda}\, d\omega, \qquad (0 \le \lambda < 1)$$

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where f constrained, and the power of $(z-\omega)^{-\lambda}$ is taken such that $\log(z-\omega)$ to be real when $z-\omega>0$. We remark that by taking $f(z)=z+\sum_{k=2}^{\infty}a_kz^k$ we have

$$D_z^{\lambda}f(z)=\frac{1}{\Gamma(2-\lambda)}z^{1-\lambda}+\sum_{k=2}^{\infty}\frac{\Gamma(k+1)}{\Gamma(k+1-\lambda)}a_kz^{k-\lambda}. \tag{1.1}$$

Definition 1.1. For $f(z)=\sum_{k=0}^{\infty}a_kz^k$ with $f\in\mathcal{H}(\mathbb{D})$, we define the linear operator

$$\Phi_{b,c}f(z)=\sum_{k=0}^{\infty}\frac{(b)_k}{(c)_k}a_kz^k, \qquad (z\in\mathbb{D}),$$

where b,c be complex numbers such that $c\neq 0,-1,-2,...,(b)_0=1$ for $b\neq 0$, and for each positive integer k, $(b)_k$ is defined by $(b)_k=b(b+1)(b+2)...(b+k-1)$.

This operator is known as a Carleson-Shaffer operator. It should be remarked that by specializing the parameters b,c and using (1.1) one can obtain

$$D_z^{\lambda}f(z)=\frac{z^{-\lambda}}{\Gamma(1-\lambda)}\Phi_{1,1-\lambda}f(z), \qquad (z\in\mathbb{D},0\leq\lambda<1), \tag{1.2}$$

where $f\in\mathcal{A}$.

We use $dA(z)$ to denote area measure on $\mathbb{D}$ and set $dA_\alpha(z)=(\alpha+1)(1-|z|^2)^\alpha dA(z)$, where $\alpha>-1$.

Definition 1.2. For $1<p<\infty$ and $\alpha\geq 1$, the analytic Besov space $\mathcal{B}_\alpha^p$ is defined as the set of all $g\in\mathcal{H}(\mathbb{D})$ such that

$$\|g\|_{\mathcal{B}_\alpha^p}:=|g(0)|+\left((p-1)\int_{\mathbb{D}}(1-|z|^2)^{p-2}|g'(z)|^p dA_\alpha(z)\right)^{\frac{1}{p}}<\infty.$$

We let $\mathcal{B}^p=\mathcal{B}_1^p$, for simplicity. Also we note that by taking $p=2$ in the Besov space $\mathcal{B}^p$, we reach the Dirichlet space $\mathcal{D}$.

Many authors have studied properties of the above mentioned classes. But till we know that, there is a few papers on acting operators on the spaces (see [1, 2]). This is natural to ask that what conditions on the parameters of b,c garantiy that for $f\in\mathbb{B}$ $(\mathcal{B}_\alpha^p)$ we have $\Phi_{b,c}f\in\mathbb{B}$ $(\mathcal{B}_\alpha^p)$ and otherwise. In this paper we try to give answere to this question.

2. Main results

Theorem 2.1. *Suppose that b,c be complex numbers such that $0<Rec<Reb$. Also let $f(z)=\sum_{k=0}^{\infty}a_kz^k\in\mathcal{H}(\mathbb{D})$ and $\Phi_{b,c}f\in\mathbb{B}$. Then $f\in\mathcal{B}_\alpha^p$ where $1<p<\infty$ and $\alpha>1$.*

Proof. We denote $\Phi_{b,c}f$ by Φ in this proof. Let $\lambda := \|\Phi\|_B$. By the hypotheses of the theorem we conclude that

$$\begin{aligned}\int_0^1 (1-t)^{b-c-1}t^{c-1}\Phi(tz)dt &= \frac{\Gamma(c)}{\Gamma(b)}\int_0^1 (1-t)^{b-c-1}t^{c-1}\sum_{k=0}^{\infty}\frac{\Gamma(b+k)}{\Gamma(c+k)}a_k t^k z^k dt \\ &= \frac{\Gamma(c)\Gamma(b-c)}{\Gamma(b)}f(z), \qquad (z\in\mathbb{D}).\end{aligned}$$

Therefore

$$|f'(z)| \le \frac{|\Gamma(b)|}{|\Gamma(c)||\Gamma(b-c)|}\int_0^1 (1-t)^{Re(b-c)-1}t^{Rec}|\Phi'(tz)|dt, \qquad (z\in\mathbb{D}). \tag{2.1}$$

By putting

$$M := \int_0^1 (1-t)^{Re(b-c)-1}t^{Rec}dt,$$

and relation (2.1), we have

$$\begin{aligned}&\int_{\mathbb{D}}(1-|z|^2)^{p-2}|f'(z)|^p dA_\alpha(z) \\ &\le \frac{|\Gamma(b)|^p(\alpha+1)}{|\Gamma(c)|^p|\Gamma(b-c)|^p}\lambda^p M^p\int_{\mathbb{D}}(1-|z|^2)^{\alpha-2}dA(z) < \infty,\end{aligned}$$

since $\alpha > 1$ and so $f \in \mathcal{B}^p_\alpha$. ☐

Corollary 2.2. *Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in \mathcal{A}$ and $z^\lambda D_z^\lambda f(z) \in \mathbb{B}$ where $0 \le \lambda < 1$. Then $f \in \mathcal{B}^p_\alpha$ where $1 < p < \infty$ and $\alpha > 1$.*

Theorem 2.3. *Suppose $1 < p < \infty, 0 < b < c-1$ and $\{a_k\}$ be bounded sequence in $\mathbb{C}$. Let $f(z) = \sum_{k=1}^{\infty} a_k z^k$ and $g(z) = \sum_{k=1}^{\infty}\frac{a_k}{k^{c-b}}z^k$. Then $g \in \mathcal{B}^p_\alpha$ if and only if $\Phi_{b,c}f \in \mathcal{B}^p_\alpha$ where $\alpha > 1$.*

Proof. Suppose $0 < b < c-1$. From Stirlings formula, we have the following asymptotic expansion for the gamma function ($|\arg z| \le \pi - \epsilon, \epsilon > 0$):

$$\Gamma(z) \approx e^{-z}z^z\sqrt{\frac{2\pi}{z}}\left(1 + \frac{1}{12z} + \frac{1}{288z^2} + ...\right) \qquad (z\to\infty).$$

Hence we obtain (with complex calculations):

$$\frac{\Gamma(b+k)}{\Gamma(c+k)} \approx k^{b-c}\sum_{i=0}^{\infty}A_i k^{-i} \qquad (k\to\infty) \tag{2.2}$$

where the A_i are constants depending on b, c with $A_0 = 1$. Now, suppose that $g(z) = \sum_{k=1}^{\infty} \frac{a_k}{k^{c-b}} z^k \in \mathcal{B}^p_\alpha$. From (2.2) we obtain

$$\Phi_{b,c} f(z) \approx \frac{\Gamma(c)}{\Gamma(b)} \sum_{k=1}^{\infty} k^{b-c} \left\{ \sum_{i=0}^{\infty} A_i k^{-i} \right\} a_k z^k,$$

and so

$$|\Phi'_{b,c} f(z)| \leq \frac{\Gamma(c)}{\Gamma(b)} |g'(z)| + \frac{\Gamma(c)}{\Gamma(b)} \sum_{k=1}^{\infty} \left| k^{b-c+1} O\left(\frac{1}{k}\right) a_k z^{k-1} \right|, \qquad (z \in \mathbb{D}). \tag{2.3}$$

Since $a_k = O(1)$ and $c - b > 1$, we have

$$\sum_{k=1}^{\infty} \left| k^{b-c+1} O\left(\frac{1}{k}\right) a_k z^{k-1} \right| \leq \max_{k \in \mathbb{N}} |a_k| \sum_{k=1}^{\infty} \left| O\left(\frac{1}{k^{c-b}}\right) \right| < \infty$$

and then, there exists a constant number N such that

$$\sum_{k=1}^{\infty} \left| k^{b-c+1} O\left(\frac{1}{k}\right) a_k z^{k-1} \right| < N. \tag{2.4}$$

Therefore from (2.3) and (2.4),

$$\begin{aligned} \int_{\mathbb{D}} \frac{|\Phi'_{b,c} f(z)|^p}{(1-|z|^2)^{2-p}} dA_\alpha(z) &\leq 2^{p-1} \frac{\Gamma(c)^p}{\Gamma(b)^p} \int_{\mathbb{D}} \frac{|g'(z)|^p}{(1-|z|^2)^{2-p}} dA_\alpha(z) \\ &\quad + 2^{p-1} \frac{\Gamma(c)^p}{\Gamma(b)^p} N^p (\alpha+1) \int_{\mathbb{D}} (1-|z|^2)^{p+\alpha-2} dA(z) \\ &< \infty, \end{aligned}$$

since $g \in \mathcal{B}^p_\alpha$ and $p + \alpha - 2 > -1$. Therefore $\Phi_{b,c} f \in \mathcal{B}^p_\alpha$.

Conversely, we can prove this part in a similar manner as the proof of part (1).

□

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