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A CORRESPONDENCE FOR UNBOUNDED REGULAR OPERATORS ON HILBERT C*-MODULES

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Abstract. Let E be a Hilbert C*-module over a fix C*-algebra and let B be a non-degenerate C*-subalgebra of $\mathcal{L}(E)$. In this paper we study the embedding of $\mathcal{R}(B)$ into $\mathcal{R}(E)$. The inclusion map $i : \mathcal{K}(E) \to \mathcal{L}(E)$ is a non-degenerate $*$-homomorphism which induces the $*$-bijection $\mathcal{R}(\mathcal{K}(E)) \to \mathcal{R}(E); t \mapsto \tilde{t} = \pi(t)$.

1. Introduction

Hilbert C*-modules are essentially objects like Hilbert spaces, except that the inner product, instead of being complex-valued, takes its values in a C*-algebra. Although Hilbert C*-modules behave like Hilbert spaces in some ways, some fundamental Hilbert space properties like Pythagoras' equality, self-duality, and even decomposition into orthogonal complements do not hold. A (right) *pre-Hilbert C*-module* over a C*-algebra A is a right A-module E equipped with an A-valued inner product $\langle \cdot, \cdot \rangle : E \times E \to A\,,\ (x, y) \mapsto \langle x, y \rangle$, which is A-linear in the second variable y and has the properties:

$$\langle x, y \rangle = \langle y, x \rangle^*, \quad \langle x, x \rangle \geq 0 \ \text{ with equality only when } x = 0.$$

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A pre-Hilbert A-module E is called a *Hilbert A-module* if E is a Banach space with respect to the norm $\|x\| = \|\langle x,x\rangle\|^{1/2}$. A Hilbert A-submodule W of a Hilbert A-module E is an orthogonal summand if $W \oplus W^{\perp} = E$, where $W^{\perp}$ denotes the orthogonal complement of W in X. We denote by $\mathcal{L}(E)$ the C*-algebra of all adjointable operators on E, i.e., all A-linear maps $t : E \to E$ such that there exists $t^* : E \to E$ with the property $\langle tx, y\rangle = \langle x, t^*y\rangle$ for all $x, y \in X$. A bounded adjointable operator $v \in \mathcal{L}(E)$ is called a *partial isometry* if $vv^*v = v$, see [**?**] for some equivalent conditions. For the basic theory of Hilbert C*-modules we refer to the books [3].

An unbounded regular operator on a Hilbert C*-module is an analogue of a closed operator on a Hilbert space. Let us quickly recall the definition. A densely defined closed A-linear map $t : Dom(t) \subseteq E \to E$ is called *regular* if it is adjointable and the operator $1 + t^*t$ has a dense range. Indeed, a densely defined operator t with a densely defined adjoint operator t^* is regular if and only if its graph is orthogonally complemented in $E \oplus E$ (see e.g. [1, 3]). We denote the set of all regular operators on E by $\mathcal{R}(E)$. If t is regular then t^* is regular and $t = t^{**}$, moreover t^*t is regular and selfadjoint. Define $q_t = (1 + t^*t)^{-1/2}$ and $f_t = tq_t$, then $Ran(q_t) = Dom(t)$, $0 \le q_t = (1 - f_t^* f_t)^{1/2} \le 1$ in $\mathcal{L}(E)$ and $f_t \in \mathcal{L}(E)$ [3, (10.4)]. The bounded operator f_t is called the bounded transform of regular operator t. According to [3, Theorem 10.4], the map $t \to f_t$ defines an adjoint-preserving bijection

$$\mathcal{R}(E) \to \{f \in \mathcal{L}(E) : \|f\| \le 1 \text{ and } Ran(1 - f^*f) \text{ is dense in } E\}.$$

Consider $t \in \mathcal{L}(E)$, then t is regular and $\|f_t\| < 1$. Consider $t \in \mathcal{R}(E)$. Then t belongs to $\mathcal{L}(E) \Leftrightarrow D(t) = E \Leftrightarrow t$ is bounded $\Leftrightarrow \|f_t\| < 1$. The space $\mathcal{R}(E)$ from a topological point of view is studied in [4]. Very often there are interesting relationships between regular operators and their bounded transforms. In fact, for a regular operator t, some properties transfer to its bounded transform F_t, and vice versa. Suppose $t \in \mathcal{R}(E)$ is a regular operator, then t is called *normal* iff $Dom(t) = Dom(t^*)$ and $\langle tx, tx\rangle = \langle t^*x, t^*x\rangle$ for all $x \in Dom(t)$. The operator t is called *selfadjoint* iff $t^* = t$ and t is called *positive* iff t is normal and $\langle tx, x\rangle \ge 0$ for all $x \in Dom(t)$. In particular, a regular operator t is normal (resp., selfadjoint, positive) iff its bounded transform f_t is normal (resp., selfadjoint, positive). Moreover, both t and f_t have the same range and the same kernel.

2. Main results

Let A, B, C be C*-algebras, such that A is an ideal in B, and let E be a Hilbert C*-module. Suppose that $\alpha : A \to \mathcal{L}(E)$ is a non-degenerate $*$-homomorphism. It is well known that α can be extended uniquely to a $*$-homomorphism $\tilde{\alpha} : B \to \mathcal{L}(E)$. If α is injective and A is essential in B then $\tilde{\alpha}$ is injective [3, Proposition 2.1]. In particular, the inclusion map $i : \mathcal{K}(E) \to \mathcal{L}(E)$ is non-degenerate, and the idealiser of $\mathcal{K}(E)$ is $\mathcal{L}(E)$, so i extends to a $*$-isomorphism between $M(\mathcal{K}(E))$, the multiplier of $\mathcal{K}(E)$, and $\mathcal{L}(E)$. The later fact motivates us for the following results.

Consider a C*-algebra A and define E to be the Hilbert C*-module over A such that $E = A$ as a right A-module and $\langle a, b\rangle = b^*a$ for every $a, b \in A$. Then the elements of $\mathcal{R}(E)$ are called elements affiliated with A. We write also $t\eta A$ instead of $t \in \mathcal{R}(E)$.

We fix a Hilbert C*-module E over a C*-algebra. At the same time, we will consider a non-degenerate C*-subalgebra B of $\mathcal{L}(E)$. We will look at an embedding of $\mathcal{R}(B)$ into $\mathcal{R}(E)$. Concerning the multiplier algebra, we have:

$$M(B) = \{\, x \in \mathcal{L}(E) \mid \text{for every } b \in B \text{ that } xb,\, bx \in B \,\}$$

As pointed out in [5, 6] for Hilbert spaces, we can also embed $\mathcal{R}(B)$ in $\mathcal{R}(E)$. Following the argument of Woronowicz [6], we state that a non-degenerate $*$-homomorphism can be extended to the set of affiliated elements.

Theorem 2.1. *Consider a Hilbert C*-module E over a C*-algebra A. Let B be a C*-algebra and π be a non-degenerate $*$-homomorphism from B into $\mathcal{L}(E)$. Consider an element t affiliated with B. Then there exists a unique element $s \in \mathcal{R}(E)$ such that $f_s = \pi(f_t)$ and we define $s = \pi(t)$. We have moreover that $\pi(D(t))\,E$ is a core for $\pi(t)$ and $\pi(t)(\pi(b)v) = \pi(t(b))\,v$ for every $b \in D(t)$ and $v \in E$.*

The last part of this theorem implies that $\pi(D)\,K$ is a core for $\pi(t)$ if D is a core for t and K is a dense subspace of E.

Remark 2.2. Suppose moreover that π is injective. Then the canonical extension of π to $M(B)$ is also injective. Let s and t be two elements affiliated with B. Utilizing the bounded transform f_t, then $s = t$ if and only if $\pi(s) = \pi(t)$.

The following result can be proven as [6, Theorem 1.2]. It follows easily using the bounded transform f_t.

Proposition 2.3. *Consider a Hilbert C*-module E over a C*-algebra A. Let B,C be two C*-algebras. Consider a non-degenerate $*$-homomorphism*

π from B into $M(C)$ and a non-degenerate $*$-homomorphism θ from C into $\mathcal{L}(E)$. Then $(\theta\pi)(t) = \theta(\pi(t))$ for every $t\,\eta\, B$.

Definition 2.4. Call π the inclusion of B into $\mathcal{L}(E)$, then π is a non-degenerate $*$-homomorphism from B into $\mathcal{L}(E)$. Let t be an element affiliated to B. Then we define $\tilde{t} = \pi(t)$, so $\tilde{t}$ is a regular operator on E.

Because π is injective, we know immediately that the mapping

$$\mathcal{R}(B) \to \mathcal{R}(E) : t \mapsto \tilde{t} = \pi(t)$$

is injective. We have also immediately that $\tilde{x} = x$ for every $x \in M(B)$. Looking at example 4 of [6], we have also the following result. Consider a regular operator t on E. Then there exists an element s affiliated with B such that $\tilde{s} = t \quad \Leftrightarrow$

(1) f_t belongs to $M(B)$,
(2) $(1 - f_t^* f_t)^{\frac{1}{2}} B$ is dense in B.

If there exists such an s, we have immediately that $f_t = f_s$, so f_t will certainly satisfy the two mentioned conditions. If f_t satisfies these two conditions, there exists an element s affiliated with B such that $f_s = f_t$. So we have that $f_{\tilde{s}} = f_s = f_t$ which implies that $\tilde{s} = t$. This implies immediately the following result.

Theorem 2.5. *Consider a Hilbert C^*-module over E over a C^*-algebra A. Then the mapping $\mathcal{R}(\mathcal{K}(E)) \to \mathcal{R}(E) : t \mapsto \tilde{t} = \pi(t)$ is a $*$-bijection.*

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