The Extended Abstracts of
The 24th Seminar on Mathematical Analysis and its Applications
26-27 May, 2021, Imam Khomeini International University, Qazvin, Iran

THE NUMERICAL RANGE OF INTERVAL MATRICES

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Abstract. In this paper, we introduce the concept of the numerical range of a square interval matrix. We prove that this numerical range is always compact, but, unlike classical numerical range, the numerical range of an interval matrix is not always convex.

1. Introduction

Let A be a (bounded linear) operator on a complex Hilbert space $\mathcal{H}$. The *numerical range* of A is the set

$$W(A) := \{\langle Ax, x\rangle : x \in \mathcal{H}, \|x\| = 1\}$$

in the complex plane, where $\langle ., .\rangle$ denotes the inner product in $\mathcal{H}$. In other words, $W(A)$ is the image of the unit sphere $\{x \in \mathcal{H} : \|x\| = 1\}$ of $\mathcal{H}$ under the (bounded) quadratic form $x \mapsto \langle Ax, x\rangle$. One of the basic properties of the numerical range is that it contains all the eigenvalues of a matrix. Two other important properties of the numerical range which we discuss them in this article are convexity (Toeplitz-Hausdorff theorem) and compactness [2]. In this paper, we extend the concept of the numerical range to interval matrices.

1991 *Mathematics Subject Classification.* Primary 47A12; Secondary 65G30.

Key words and phrases. numerical range; interval matrix; interval norm; convexity; compactness.

Interval arithmetic is an approach that can be used to bound measurement errors, rounding and truncation errors in mathematical computations. As an area of numerical analysis, it has several applications in different branches of science and engineering. One of the advantages of interval analysis is its ability to compute bounds on the range of functions. By extending interval arithmetic to vectors and matrices, it became a useful tool to find reliable and guaranteed solutions to matrix equations and optimization problems. For more details see [6, 8, 7].

The interval eigenvalue problem, which is the task of finding intervals that contain all the possible eigenvalues of the matrices in an interval matrix, has an important role in many engineering problems. See for example [4, 3] for further information about the efforts for solving interval eigenvalue problem. Due to the spectral inclusion property of the numerical range, the numerical range of interval matrices, contains all the eigenvalues of the matrices in the interval. Therefore, finding the numerical range of interval matrices can help us to solve the interval eigenvalue problems.

The paper is organized as follows. In Section 2, we consider some basic definitions, notations and properties of interval arithmetic. In Section 3, we define the concept of the numerical range for real interval matrices, show its relation with interval norm, prove its compactness and give a counterexample that demonstrates the numerical range of interval matrices is not always convex.

2. Definitions, notations and basic facts

In this section we give some basic definitions, notations and properties of interval arithmetic.

Definition 2.1. [5] An interval matrix is defined as

$$\mathbf{A} := \left[\underline{A}, \overline{A}\right] = \{A \in \mathbb{R}^{m\times n} : \underline{A} \leq A \leq \overline{A}\},$$

where $\underline{A} = (\underline{a}_{ij})$, $\overline{A} = (\overline{a}_{ij}) \in \mathbb{R}^{m\times n}$, $\underline{A} \leq \overline{A}$, are given *point matrices*, where the concept of point matrix is used to mention any conventional matrix.

Throughout this paper, the notation $Conv(E)$ is used for the convex hull of the set E, the set of $m \times n$ real interval matrices is denoted by $\mathbb{IR}^{m\times n}$ and the comparison relations $\leq$ and $\geq$ are interpreted component-wise.

Definition 2.2. (Interval Matrix Norm) [1] A function $|||\cdot||| : \mathbb{IR}^{m\times n} \to \mathbb{R}$ is called an interval matrix norm in $\mathbb{IR}^{m\times n}$ if for each $\mathbf{A}, \mathbf{B} \in \mathbb{IR}^{m\times n}$ and $\alpha \in \mathbb{R}$ it satisfies the following properties:

(1) $|||\mathbf{A}||| \geq 0$, and $|||\mathbf{A}||| = 0$ if and only if $\mathbf{A} = [0, 0]$;
(2) $|||\mathbf{A} + \mathbf{B}||| \leq |||\mathbf{A}||| + |||\mathbf{B}|||$;
(3) $|||\alpha\mathbf{A}||| = |\alpha||||\mathbf{A}|||$,

where $|\cdot|$ is the conventional absolute value.

The following theorem shows how to construct interval matrix norms from point matrix norms.

Theorem 2.3. [1] *For any point matrix norm $\|\cdot\|$ in $\mathbb{R}^{m\times n}$, the function $|||\cdot||| : \mathbb{IR}^{m\times n} \to \mathbb{R}$ defined by*

$$|||\boldsymbol{A}||| = \sup\{\|\beta\| : \beta \in \boldsymbol{A}\},$$

is an interval matrix norm in $\mathbb{IR}^{m\times n}$.

3. The numerical range of $n \times n$ interval matrices

In this section after defining the concept of the numerical range and the numerical radius of interval matrices, we show that like the conventional case it is compact. Also, by an example we will show that in general we do not have the property of convexity.

Definition 3.1. Let $\mathbf{A} \in \mathbb{IR}^{n\times n}$. We define the numerical range of $\mathbf{A}$ by

$$W(\mathbf{A}) := \bigcup_{A\in\mathbf{A}} W(A),$$

where $W(A)$ is the conventional numerical range of the point matrix $A \in \mathbb{R}^{n\times n}$. Also, the numerical radius of $\mathbf{A}$ is defined by

$$r(\mathbf{A}) := \sup\{r(A) : A \in \mathbf{A}\},$$

where $r(A)$ is the conventional numerical radius of A.

Lemma 3.2. *Suppose that $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$. Then*

$$W(\boldsymbol{A}) \subseteq \{\mu \in \mathbb{C} : |\mu| \leq |||\boldsymbol{A}|||\}.$$

Lemma 3.3. *Let $\boldsymbol{A} = \left[\underline{A}, \overline{A}\right] \in \mathbb{IR}^{n\times n}$ and $\underline{A} \geq 0$. Then*

$$W(\boldsymbol{A}) \subseteq \{\mu \in \mathbb{C} : |\mu| \leq \|\overline{A}\|_\infty\}.$$

Theorem 3.4. *Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$. Then $W(\boldsymbol{A})$ is a compact set.*

In the following example we will show that, unlike the conventional numerical range, the numerical range of interval matrices is not necessarily convex.

Example 3.5. Assume that $\mathbf{A} = \left\{ A_\alpha = \begin{bmatrix} 11 & 5 \\ \alpha & 0 \end{bmatrix} : \alpha \in [1,6] \right\}$. It is clear that the eigenvalues of A_α are

$$\lambda_{1,2} = \frac{11 \pm \sqrt{121 + 20\alpha}}{2}.$$

The numerical range of $A_\alpha, W(A_\alpha)$, is an ellipse with minor and major axis of $|\alpha - 5|$ and $\sqrt{\alpha^2 + 10\alpha + 146}$, respectively (see [2]). It is easy to check that the points $(x_1, y_1) = (\frac{11}{2}(1-\sqrt{2}), 0)$ and $(x_2, y_2) = (\frac{11}{2}, 2)$ are in $W(A_\alpha)$, for $\alpha = 6$ and $\alpha = 1$, respectively. We will show that $(x^*, y^*) = (-1, \frac{22-13\sqrt{2}}{11})$, a point on the straight line segment joining (x_1, y_1) and (x_2, y_2), is not in $W(\mathbf{A})$.

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