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THE STABILITY OF THE CUBIC B-SPLINE COLLOCATION METHOD FOR STOCHASTIC TIME-FRACTIONAL ADVECTION-DIFFUSION EQUATION

A. YAZDANI CHERATI, Z. AZIMI

Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran
yazdani@umz.ac.ir

Abstract. The ultimate goal of this performs study is to provide proof of the stability of an approach based on cubic B-spline collocation methods (CBSCM) for solving the time-fractional stochastic advection-diffusion equation (TFSADE). We prove that the stability. We prove that the proposed scheme is unconditionally stable.

1. Introduction

Recently, finding a solution for a class of fractional differential equations involving Brownian motion is highly important, because this type of equation is rarely be solved due to randomness, and the analysis of differential equations involving random coefficients gives us more details of the phenomenon behavior.

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We originally intend to obtain the numerical solution of the following stochastic equation:

$$\begin{aligned} &D_t^{\alpha}u(x,t) + Lu(x,t) = f(x,t), \quad x \in (a,b), \quad t \in (0,T], \\ &Lu(x,t) = \sigma_1 \frac{\partial u(x,t)}{\partial x} - (\sigma_2 + \sigma_3 \dot{B}(t)) \frac{\partial^2 u(x,t)}{\partial x^2}, \end{aligned} \tag{1.1}$$

with the following initial and boundary conditions

$$u(x,0) = g(x), \quad x \in [a,b], \tag{1.2}$$

$$u(a,t) = u(b,t) = 0, \quad t \in (0,T], \tag{1.3}$$

where σ_1 is the coefficient of advection, and σ_2 and σ_3 are the coefficients of diffusion terms. $g(x)$ is a continuous function. The source function $f(x,t)$ is a sufficiently smooth function. Here, $L$is a linier spatial derivative operator and $0 \le \alpha < 1$ is order fractional coefficient of the equation. Also, the phrase $\dot{B}(t) = \frac{dB(t)}{dt}$ is white noise where $B(t)$ is a Brownian motion. For discretization of $B(t)$, we set $t = t_j$ and let $B_j = B(t_j)$.

2. Numerical Scheme

First we consider two arbitrary constants $M, N \in \mathbb{N}$. We assume

$$\begin{aligned} &a = x_0 < x_1 < \cdots < x_M = b, \quad x_i = a + i(\frac{b-a}{M}), (i = 0,1,2,\cdots,M) \\ &0 = t_0 < t_1 < \cdots < t_N = T, \quad t_k = k(\frac{T}{N}), (k = 0,1,2,\cdots,N). \end{aligned}$$

are uniform partition in the solution domain $[a,b]$ and $[0,T]$, respectively.

Now let $B_m(x)$ for $m = -1, \cdots, M+1$ be the cubic trigonometric B-spline function in the uniform partition on $[a,b]$ that can be defined as follows

$$B_m(x) = \frac{1}{h^3} \begin{cases} (x - x_{m-2})^3, & x \in [x_{m-2}, x_{m-1}], \\ h^3 + 3h^2(x - x_{m-1}) + 3h(x - x_{m-1})^2 - 3(x - x_{m-1})^3, & x \in [x_{m-1}, x_m], \\ h^3 + 3h^2(x_{m+1} - x) + 3h(x_{m+1} - x)^2 - 3(x_{m+1} - x)^3, & x \in [x_m, x_{m+1}], \\ (x_{m+2} - x)^3, & x \in [x_{m+1}, x_{m+2}], \\ 0, & o.w. \end{cases} \tag{2.1}$$

It is obvious that the support of the cubic trigonometric B-spline $B_m(x)$ and its derivative is $[x_{m-2}, x_{m+2}]$.

Let $u(x,t)$ and $U(x,t)$ are the analytical and numerical solutions of the differential equation (1.1), respectively. According to the collocation method, the numerical solution can be approximated as

$$u(x,t) \simeq U(x,t) = \sum_{m=-1}^{M+1} \delta_m(t) B_m^3(x), \tag{2.2}$$

and the coefficients $\delta_m(t)$ are to be determined by the numerical scheme proposed in this paper.

The matrix form of Equation (2.3) is as follows

$$A\boldsymbol{\delta}^{k+1} = B\left(b_k\boldsymbol{\delta}^0 + \sum_{j=0}^{k-1}(b_j - b_{j+1})\boldsymbol{\delta}^{k-j}\right) + \boldsymbol{f}^{k+1}, \quad k = 0, 1, \cdots, n-1, \tag{2.3}$$

where $\boldsymbol{\delta}^k = [\delta_{-1}^k, \delta_0^k, \delta_1^k, \cdots, \delta_N^k, \delta_{N+1}^k]^T$ is the unknown parameters, A and B are the coefficients matrices and $\boldsymbol{f}^k = [f_0^k, \cdots, f_N^k]^T$ and the matrices A and B are as follows

$$A = \begin{pmatrix} \frac{12\sigma_1}{h} + \frac{36\sigma_2}{h^2} & & & & & & \\ a_1 & a_2 & a_3 & & & & \\ & a_1 & a_2 & a_3 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & a_1 & a_2 & a_3 & \\ & & & & a_1 & a_2 & a_3 \\ & & & & & & \frac{12\sigma_1}{h} + \frac{36\sigma_2}{h^2} \end{pmatrix},$$

where $r - \frac{3\sigma_1}{h} - \frac{6\theta_j}{h^2} := a_1^j$, $4r + \frac{12\theta_j}{h^2} := a_2^j$ and $r + \frac{3\sigma_1}{h} - \frac{6\theta_j}{h^2} := a_3^j$. $\dot{B} \simeq \frac{B(t_j) - B(t_{j-1})}{\tau} := \zeta_j$ for $j = 1, \cdots, N$, and $\sigma_2 + \sigma_3\zeta_j := \theta_j$.

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & & & & \\ r & 4r & r & 0 & & & & \\ 0 & r & 4r & r & & & & \\ & & \ddots & \ddots & \ddots & & & \\ & & & & r & 4r & r & 0 \\ & & & & 0 & r & 4r & r \\ & & & & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{f}^{k+1} = \begin{pmatrix} 0 \\ \vdots \\ f^{k+1} \\ \vdots \\ 0 \end{pmatrix}.$$

The matrix A in $(N+1)\times(N+1)$ dimensions is a symmetric positive definite matrix.

3. The stability of the method

Theorem 3.1. *The numerical scheme (2.3) for solving the initial and boundary value problem (1.1)-(1.2) is unconditionally stable.*

Proof. We used the Von-Neumann processes and suppose that $f(x,t) = 0$. Since the error of the method is only related to the time parameters δ_m^k, denoting $\boldsymbol{e}^k := \boldsymbol{\delta}^{k+1} - \boldsymbol{\delta}^k$ as the error of scheme at time level k, the numerical scheme (2.3) can be rewritten as

$$(r - \frac{3\sigma_1}{h} - \frac{6\theta_j}{h^2})e_{m-1}^{k+1} + (4r + \frac{12\theta_j}{h^2})e_m^{k+1} + (r + \frac{3\sigma_1}{h} - \frac{6\theta_j}{h^2})e_{m+1}^{k+1} = re_{m-1}^k + 4re_m^k + re_{m+1}^k$$
$$- \sum_{s=1}^{k} b_s\{r[e_{m-1}^{k-s+1} - e_{m-1}^{k-s}] + 4r[e_m^{k-s+1} - e_m^{k-s}] + r[e_{m+1}^{k-s+1} - e_{m+1}^{k-s}]\}. \tag{3.1}$$

Then substituting the Fourier mode $e_m^k = v^k e^{im\rho}(i := \sqrt{-1})$ into (3.1) results

$$v^{k+1}\{(r - \frac{3\sigma_1}{h} - \frac{6\theta_j}{h^2})e^{-i\rho} + (4r + \frac{12\theta_j}{h^2}) + (r + \frac{3\sigma_1}{h} - \frac{6\theta_j}{h^2})e^{i\rho} = v^k\{re^{-i\rho} + 4r + re^{i\rho}\}$$
$$- \sum_{s=1}^{k} b_s\{(v^{k-s+1} - v^{k-s}) \times (re^{-i\rho} + 4r + re^{i\rho}),$$

or $v^{k+1} = Q\{b_k v^0 + \sum_{s=0}^{k-1}(b_s - b_{s+1})v^{k-s}\}$, where

$$Q = \frac{2r(1 + 2\cos^2(\frac{\rho}{2}))}{2r(1 + 2\cos^2(\frac{\rho}{2})) + \frac{24\sigma_2}{h^2}\sin^2(\frac{\rho}{2}) + i\frac{6\sigma_1}{h}\sin(\rho)}.$$

It is very clear that $|Q|^2 \leq 1$. For $k = 0$, (**??**) we have $v^1 = Qb_0v^0$, and can be written $|v^1| = |Q||b_0||v^0| \leq |v^0|$. Let $|v^j| \leq |v^0|$, $j = 1, 2, ..., k$. We have,

$$|v^{k+1}| \leq b_k|v^0| + \sum_{s=0}^{k-1}(b_s - b_{s+1})|v^{k-s}| \leq b_s|v^0| + \sum_{s=0}^{k-1}(b_s - b_{s+1})|v^0|$$
$$= |v^0|(b_s + \sum_{s=0}^{k-1}(b_s - b_{s+1})) \leq |v^0|.$$

This relation shows that $|e^{k+1}| \leq |e^0|$, and as a result, the method is unconditionally stable. □

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