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SOME RESULTS ON OSTROWSKI'S INEQUALITY

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Abstract. In this paper, we establish Ostrowski's type inequality for uniformly s-convex functions. Also, we obtain some new inequalities of Ostrowski's type for functions whose derivatives in absolute value are the class of uniformly s-convex.

1. Introduction

In 1928 Ostrowski proved the following result:
IF $f : I \to \mathbb{R}$ is continous on (a, b) and $f' : I \to \mathbb{R}$ is bounded on (a, b) such that $||f'||_\infty < \infty$ then

$$|f(x) - \frac{1}{b-a}\int_a^b f(t)dt| \leq [\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2}](b-a)||f'||_\infty$$

for all $x \in (a, b)$. The constant $\frac{1}{4}$ in above inequality is the best. Because of the attractiveness of the inequality topic, in recent years, a lot of researchers have improved the Ostrowski and other inequality to other functions (see[1], [3], [4], [5]).
In this section, we consider the basic concepts and results, which are needed to obtain our main results.

In [[2], Definition 10.5], the class of uniformly convex functions is defined as follows and we generalize this definition to uniformly convex functions in the following.

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Definition 1.1. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Then f is called uniformly s-convex function with modulus $\psi : [0, +\infty) \to [0, +\infty]$ if ψ is increasing, ψ vanishes only at 0, and

$$f(tx + (1-t)y) + t^s(1-t)\psi(|x-y|) \leq t^s f(x) + (1-t)^s f(y), \quad (1.1)$$

for each $x, y \in [0, +\infty)$ and $t \in [0, 1]$. Furthermore, if $s = 1$, then f is called uniformly convex.

Example 1.2. ([2]) In view of the equality,

$$(tx + (1-t)y)^2 + t(1-t)(x-y)^2 = tx^2 + (1-t)y^2,$$

for all $t \in (0, 1)$ and $x, y \in \mathbb{R}$, the function $f(t) = t^2$ for $t \in \mathbb{R}$ is uniformly convex with $s = 1$ and modulus $\psi(t) = t^2$ for all $t \geq 0$.

In [1], Alomari et al. proved the following inequality of Ostrowski type for functions whose derivative in absolute value are s-convex in the second sense.

Lemma 1.3. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I^o and $a, b \in I$ with $a < b$. If $f' \in L[a, b]$, then the following equality holds:*

$$\begin{aligned} f(x) - \frac{1}{b-a}\int_a^b f(t)dt = &\frac{(x-a)^2}{b-a}\int_0^1 tf'(tx + (1-t)a)dt \\ &- \frac{(b-x)^2}{b-a}\int_0^1 tf'(tx + (1-t)b)dt \end{aligned}$$

for each $x \in [a, b]$.

2. Main results

2.1. **Ostrowski type inequalities.**

Theorem 2.1. *Let $f : I \subset [0, +\infty) \to [0, +\infty)$ be a differentiable mapping on I^o such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'|$ is uniformly s-convex on $[a, b]$ for some fixed $s \in (0, 1]$ and $|f'(x)| \leq M, x \in [a, b]$, then the following inequality holds:*

$$\begin{aligned} |f(x) - \frac{1}{b-a}\int_a^b f(t)dt| \leq &\frac{M}{b-a}[\frac{(x-a)^2 + (x-b)^2}{s+1}] \\ &- \frac{1}{b-a}[\frac{(x-a)^2\psi(|x-a|) + (x-b)^2\psi(|x-b|)}{(s+1)(s+2)}]. \end{aligned}$$

Proof. In view of Lemma 1.3 and uniformly s-convexity of $|f'|$, one has

$$|f(x) - \frac{1}{b-a}\int_a^b f(t)dt| \leq$$

$$\frac{(x-a)^2}{b-a}\int_0^1 t|f'(tx+(1-t)a)|dt + \frac{(b-x)^2}{b-a}\int_0^1 t|f'(tx+(1-t)b)|dt$$

$$\leq \frac{(x-a)^2}{b-a}\int_0^1 t[t^s|f'(x)| + (1-t)^s|f'(a)| - t^s(1-t)\psi(|x-a|)]dt$$

$$+ \frac{(x-b)^2}{b-a}\int_0^1 t[t^s|f'(x)| + (1-t)^s|f'(b)| - t^s(1-t)\psi(|x-b|)]dt$$

$$\leq \frac{(x-a)^2}{b-a}[|f'(x)|\int_0^1 t^{s+1}dt + |f'(a)|\int_0^1 t(1-t)^s dt$$

$$- \int_0^1 t^{s+1}(1-t)\psi(|x-a|)dt]$$

$$+ \frac{(x-b)^2}{b-a}[|f'(x)|\int_0^1 t^{s+1}dt + |f'(b)|\int_0^1 t(1-t)^s dt$$

$$- \int_0^1 t^{s+1}(1-t)\psi(|x-b|)dt]$$

$$\leq \frac{(x-a)^2}{b-a}[\frac{|f'(x)|}{s+2} + \frac{\Gamma(2)\Gamma(s+1)}{\Gamma(s+3)}|f'(a)| - \frac{\Gamma(s+2)\Gamma(2)}{\Gamma(s+4)}\psi(|x-a|)]$$

$$+ \frac{(x-b)^2}{b-a}[\frac{|f'(x)|}{s+2} + \frac{\Gamma(2)\Gamma(s+1)}{\Gamma(s+3)}|f'(b)| - \frac{\Gamma(s+2)\Gamma(2)}{\Gamma(s+4)}\psi(|x-b|)]$$

$$\leq \frac{(x-a)^2}{b-a}[\frac{|f'(x)|}{s+2} + \frac{|f'(a)|}{(s+1)(s+2)} - \frac{\psi(|x-a|)}{(s+3)(s+2)}]$$

$$+ \frac{(x-b)^2}{b-a}[\frac{|f'(x)|}{s+2} + \frac{|f'(b)|}{(s+1)(s+2)} - \frac{\psi(|x-b|)}{(s+3)(s+2)}]$$

$$\leq \frac{(x-a)^2}{b-a}[\frac{M}{s+1} - \frac{\psi(|x-a|)}{(s+3)(s+2)}] + \frac{(x-b)^2}{b-a}[\frac{M}{s+1} - \frac{\psi(|x-a|)}{(s+3)(s+2)}]$$

$$\leq \frac{M}{b-a}[\frac{(x-a)^2 + (x-b)^2}{s+1}] - \frac{1}{b-a}[\frac{(x-a)^2\psi(|x-a|) + (x-b)^2\psi(|x-b|)}{(s+3)(s+2)}]$$

□

Remark 2.2. In Theorem 2.3, if $s = 1$, then

$$|f(x) - \frac{1}{b-a}\int_a^b f(t)dt| \leq \frac{M}{b-a}[\frac{(x-a)^2 + (x-b)^2}{2}]$$
$$- \frac{1}{b-a}[\frac{(x-a)^2\psi(|x-a|) + (x-b)^2\psi(|x-b|)}{12}].$$

New inequalities of Ostrowski's type for uniformly s-convex functions as follows:

Theorem 2.3. *Let $f : I \subset [0,+\infty) \to [0,+\infty)$ be a differentiable mapping on I^o such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is uniformly s-convex on $[a,b]$ for some fixed $s \in (0,1], p, q > 1, \frac{1}{p}+\frac{1}{q} = 1$ and $|f'(x)| \leq M, x \in [a,b]$, then the following inequality holds:*

$$\begin{aligned}|f(x) - \frac{1}{b-a}\int_a^b f(t)dt| \leq &\frac{(x-a)^2}{b-a}(\frac{1}{p+1})^{\frac{1}{p}}[\frac{2(s+2)M^q - \psi(|x-a|)}{(s+1)(s+2)}]^{\frac{1}{q}} \\ &+ \frac{(x-b)^2}{b-a}(\frac{1}{p+1})^{\frac{1}{p}}[\frac{2(s+2)M^q - \psi(|x-b|)}{(s+1)(s+2)}]^{\frac{1}{q}}\end{aligned}$$

Proof. By Lemma 1.3 and Hölder's inequality, we conclude

$$\begin{aligned}&|f(x) - \frac{1}{b-a}\int_a^b f(t)dt| \leq \\ &\frac{(x-a)^2}{b-a}[(\int_0^1 t^p dt)^{\frac{1}{p}}(\int_0^1 |f'(tx+(1-t)a)|^q dt)^{\frac{1}{q}} \\ &+ \frac{(x-b)^2}{b-a}[(\int_0^1 t^p dt)^{\frac{1}{p}}(\int_0^1 |f'(tx+(1-t)b)|^q dt)^{\frac{1}{q}} \\ &\leq \frac{(x-a)^2}{b-a}(\frac{1}{p+1})^{\frac{1}{p}}(\int_0^1 t^s|f'(x)|^q dt + \int_0^1 (1-t)^s|f'(a)|^q \\ &- \psi(|x-a|)\int_0^1 t^s(1-t)dt)^{\frac{1}{q}} \\ &+ \frac{(x-b)^2}{b-a}(\frac{1}{p+1})^{\frac{1}{p}}(\int_0^1 t^s|f'(x)|^q dt + \int_0^1 (1-t)^s|f'(a)|^q \\ &- \psi(|x-b|)\int_0^1 t^s(1-t)dt)^{\frac{1}{q}} \\ &\leq \frac{(x-a)^2}{b-a}(\frac{1}{p+1})^{\frac{1}{p}}(\frac{|f'(x)|^q}{s+1} + \frac{|f'(a)|^q}{s+1} - \frac{\psi(|x-a|)}{(s+1)(s+2)})^{\frac{1}{q}} \\ &+ \frac{(x-b)^2}{b-a}(\frac{1}{p+1})^{\frac{1}{p}}(\frac{|f'(x)|^q}{s+1} + \frac{|f'(b)|^q}{s+1} - \frac{\psi(|x-b|)}{(s+1)(s+2)})^{\frac{1}{q}} \\ &\leq \frac{(x-a)^2}{b-a}(\frac{1}{p+1})^{\frac{1}{p}}[\frac{2(s+2)M^q - \psi(|x-a|)}{(s+1)(s+2)}]^{\frac{1}{q}} \\ &+ \frac{(x-b)^2}{b-a}(\frac{1}{p+1})^{\frac{1}{p}}[\frac{2(s+2)M^q - \psi(|x-b|)}{(s+1)(s+2)}]^{\frac{1}{q}}.\end{aligned}$$

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