KANNAN FIXED-POINT THEOREM ON BANACH GROUPS

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Abstract. In this paper we present a fixed point theorem for Kannan type mapping in a Banach group.

1. Introduction

The fixed point theory is one of the most useful and essential tools of nonlinear analysis. The first important result in the theory of fixed point about contractive mapping is Banach theorem [1]. In 1968 Kannan [5] introduced a new type of contraction. Subrahmanyam [6] showed that a metric spaces is complete if and only if, every Kannan mapping has a fixed point. In 2018 Karapinar, by using the interpolation notion, introduced a new Kannan type contraction to maximize the rate of convergence [4].

On the other hand, group-norms have also played a role in the theory of topological groups [2, 3]. Some results on the existence and uniqueness of fixed points for Kannan mappings on normed groups and Banach group are proved in this paper. .

We begin with some basic notions which will be needed in this paper.

Definition 1.1. [2] Let $\mathcal{L}$ be a group. A norm on a group $\mathcal{L}$ is a function $\|.\| : \mathcal{L} \to \mathbb{R}$ with the following properties:

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(1) $\|w\| \geq 0,$ *for all* $w \in \mathcal{L}$;
(2) $\|w\| = \|w^{-1}\|,$ *for all* $w \in \mathcal{L}$;
(3) $\|wk\| \leq \|w\| + \|k\|,$ *for all* $w, k \in \mathcal{L}$;
(4) $\|w\| = 0$ *implies that* $w = e$.

A normed group $(\mathcal{L}, \|.\|)$ is a group $\mathcal{L}$ equipped with a norm $\|.\|$. By setting $d(w,k) := \|w^{-1}k\|$, it is easy to see that norms are in bijection with left-invariant metrics on $\mathcal{L}$.

Definition 1.2. A Banach group is a normed group $(\mathcal{L}, \|.\|)$, which is complete with respect to the metric

$$d(w,k) = \|wk^{-1}\|, \quad (w, k \in \mathcal{L}).$$

Definition 1.3. Let $(\mathcal{L}, \|.\|)$ be a normed group and $\vartheta : \mathcal{L} \to \mathcal{L}$ be a mapping. Then ϑ is called Kannan contraction if there exists $\eta \in [0, \frac{1}{2})$ such that

$$\|\vartheta(w)\vartheta(k)^{-1}\| \leq \eta \left[\|w\vartheta(w)^{-1}\| + \|k\vartheta(k)^{-1}\| \right],$$

for all $w, k \in \mathcal{L}$.

2. Main results

In this section we extend the kannan's theorem and we continue by a generalization of the definition of Kannan type contraction via interpolation notion.

Lemma 2.1. *Let* $(\mathcal{L}, \|.\|)$ *be a Banach group and* A *be a nonempty closed subset of* $\mathcal{L}$ *and let* $\psi : A \to A$ *be a mapping such that satisfying*

$$\|\psi(w)\psi(k)^{-1}\| \leq \eta \left[\|w\psi(w)^{-1}\| + \|k\psi(k)^{-1}\| \right],$$

for all $w, k \in A$ *and* $0 \leq \eta < 1$. *If for arbitrary point* $a \in A$ *there exists* $b \in A$ *such that* $\|b\psi(b)^{-1}\| \leq r_1 \|a\psi(a)^{-1}\|$ *and* $\|ba^{-1}\| \leq r_2 \|a\psi(a)^{-1}\|$, *when there exist constants* $r_1, r_2 \in \mathbb{R}$ *such that* $0 \leq r_1 < 1$ *and* $r_2 > 0$, *Then* ψ *has at least one fixed point.*

Proof. For an arbitrary element $a_0 \in A$ define a sequence $(a_n)_{n=1}^{\infty} \subset A$ such that

$$\|\psi(a_{n+1})a_{n+1}^{-1}\| \leq r_1 \|\psi(a_n)a_n^{-1}\|,$$

and

$$\|a_{n+1}a_n^{-1}\| \leq r_2 \|\psi(a_n)a_n^{-1}\|,$$

for $n = 1, 2, \ldots.$ It is easy to see that $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence. Because A is complete, there exists $c \in A$ such that $\lim\limits_{n\to\infty} a_n = c$. Then

$$\begin{aligned}
\|\psi(c)c^{-1}\| &\leq \|\psi(c)\psi(a_n)^{-1}\| + \|\psi(a_n)a_n^{-1}\| + \|a_n c^{-1}\| \\
&\leq \eta \left[\|c\psi(c)^{-1}\| + \|a_n\psi(a_n)^{-1}\| \right] + \|\psi(a_n)a_n^{-1}\| + \|a_n c^{-1}\|,
\end{aligned}$$

and

$$\begin{aligned}\|\psi(c)c^{-1}\| &\leq \frac{\eta+1}{1-\eta}\|\psi(a_n)a_n^{-1}\| + \frac{1}{1-\eta}\|a_nc^{-1}\| \\ &\leq \frac{\eta+1}{1-\eta}r_1^n\|\psi(a_0)a_0^{-1}\| + \frac{1}{1-\eta}\|a_nc^{-1}\| \to 0,\end{aligned}$$

as $n \to \infty$. So, $\psi(c) = c$. □

Theorem 2.2. *Let $(\mathcal{L}, \|.\|)$ be a Banach group and ϑ be a Kannan contraction. Then ϑ has a unique fixed point in $\mathcal{L}$.*

Proof. Let $w \in \mathcal{L}$ be an arbitrary element. Consider $k = \vartheta(w)$. Then

$$\|k\vartheta(k)^{-1}\| = \|\vartheta(w)\vartheta(k)^{-1}\| \leq \eta\left[w\vartheta(w)^{-1}\| + \|k\vartheta(k)^{-1}\|\right],$$

which implies

$$\|k\vartheta(k)^{-1}\| \leq \frac{\eta}{1-\eta}\|w\vartheta(w)^{-1}\|.$$

For arbitrary $w_0 \in \mathcal{L}$ define a sequence $(w_{n+1} = \vartheta(w_n))_{n=1}^{\infty}$. By the Lemma (2.1), this sequence is converges to z and $\vartheta(z) = z$. If w be an another fixed point of ϑ, we have

$$0 < \|zw^{-1}\| = \|\vartheta(z)\vartheta(w)^{-1}\| \leq \eta\left[\|z\vartheta(z)^{-1}\| + \|w\vartheta(w)^{-1}\|\right] = 0.$$

Therefore, ϑ has a unique fixed point.

□

Theorem 2.3. *Let $(\mathcal{L}, \|.\|)$ be a compact normed group and let $\vartheta : \mathcal{L} \to \mathcal{L}$ be a continuous Kannan nonexpansive mapping. Then ϑ has a unique fixed point.*

Proof. The function $\alpha : \mathcal{L} \to [0, \infty)$ defined by $\alpha(w) = \|w\vartheta(w)^{-1}\|$ is continuous. Since $\mathcal{L}$ is compact, there exists an element $z \in \mathcal{L}$ such that $\vartheta(z) = \inf\{\vartheta(w) : \ w \in \mathcal{L}\}$. If $\vartheta(z) \neq z$, then

$$\|\vartheta(z)\vartheta(\vartheta(z))^{-1}\| < \frac{1}{2}\left[\|z\vartheta(z)^{-1}\| + \|\vartheta(z)\vartheta(\vartheta(z))^{-1}\|\right],$$

and

$$\alpha(\vartheta(z)) = \|\vartheta(z)\vartheta(\vartheta(z))^{-1}\| < \|z\vartheta(z)^{-1}\| = \alpha(z).$$

This is a contradiction and hence, $\vartheta(z) = z$. It is obvious that z is a unique fixed point. □

Definition 2.4. Let $(\mathcal{L}, \|.\|)$ be a normed group. We say that the self-mapping $\vartheta : \mathcal{L} \to \mathcal{L}$ is an interpolative Kannan type contraction, if there exist a constant $\eta \in [0, 1)$ and $\mu \in (0, 1)$ such that

$$\|\vartheta(w)\vartheta(k)^{-1}\| \leq \eta[\|w\vartheta(w)^{-1}\|]^{\mu}.[\|k\vartheta(k)^{-1}\|]^{1-\mu}, \tag{2.1}$$

for all $w, k \in \mathcal{L}$ with $w \neq \vartheta(w)$.

Theorem 2.5. *Let $(\mathcal{L}, \|.\|)$ be a Banach group and ϑ be an interpolative Kannan type contraction.Then ϑ has a unique fixed point in $\mathcal{L}$.*

Proof. For arbitrary element $w_0 \in \mathcal{L}$ a sequence $(w_n)_{n=1}^{\infty} \subset \mathcal{L}$ be defined by $w_{n+1} = \vartheta^n(w_0)$. Without loss of generality, we assume that $w_n \neq w_{n+1}$ for each non-negative integer n. Then $\|w_n\vartheta(w_n)^{-1}\| = \|w_n w_{n+1}^{-1}\| > 0$. By taking $w = w_n$ and $k = w_{n-1}$ in (2.1) we get

$$\begin{aligned}\|w_{n+1}w_n^{-1}\| = \|\vartheta(w_n)\vartheta(w_{n-1})^{-1}\| &\leq \eta[\|w_n\vartheta(w_n)^{-1}\|]^{\mu}[\|w_{n-1}\vartheta(w_{n-1})^{-1}\|]^{1-\mu} \\ &= \eta[\|w_{n-1}w_n^{-1}\|]^{1-\mu}[\|w_n w_{n+1}\|]^{\mu},\end{aligned}$$

so

$$\|w_n w_{n+1}^{-1}\|^{1-\mu} \leq \eta[\|w_{n-1}w_n^{-1}\|]^{1-\mu}.$$

So, the sequence $\|w_{n-1}w_n^{-1}\|$ is non-increasing and non-negative. As a result, there is a non-negative constant z such that $\lim_{n\to\infty} \|w_{n-1}w_n^{-1}\| = z$. Then

$$\|w_n w_{n+1}^{-1}\| \leq \eta\|w_{n-1}w_n^{-1}\| \leq \eta^n\|w_0 w_1^{-1}\|.$$

By letting $n \to \infty$ in the inequality above, we observe that $z = 0$. By using a standard arguments based on the triangle inequality, we conclude that the sequence $(w_n)_{n=1}^{\infty}$ is a Cauchy sequence.Since $\mathcal{L}$ is a complete group, there exists $v \in \mathcal{L}$ such that $\lim_{n\to\infty} \|w_n v^{-1}\| = 0$. By substituting $w = w_n$ and $k = v$ in (2.1), we have

$$\|\vartheta(w_n)\vartheta(v)^{-1}\| \leq \eta[\|w_n\vartheta(w_n)^{-1}\|]^{\mu}.[\|v\vartheta(v)^{-1}\|]^{1-\mu}.$$

Taking $n \to \infty$ in the inequality above, we thus get $\|v\vartheta(v)^{-1}\| = 0$ and hence $v = \vartheta(v)$. It is obvious that v is a unique fixed point.

□

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