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SOME PROPERTIES OF F-HARMONIC MAPS WITH POTENTIAL

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Abstract. In this paper, $F-$harmonic maps with potential between Riemannian manifolds are studied. First, the variational formulas for these types of maps are obtained. Then, stability of $F-$harmonic maps from a Riemannian manifold into a standard unit sphere is studied.

1. Introduction

In 1964, Sampson and Eells investigated the properties of harmonic maps. They also proved the fundamental existence theorem for harmonic maps. From up to now, many scholars have done research on this topic,[3, 4]. These kind of maps have an important role in many branch of physics, mathematics and mechanics such as liquid crystal, ferromagnetic material, super conductor, etc., see [5, 6].
In [7], Ratto introduced the notion of harmonic maps with potential. Recently many research have done on this topic, Y. Chu [2]. Let H be a smooth function on a smooth manifold N and let $\phi : (M, g) \longrightarrow (N, h)$ be a smooth map between Riemannian manifolds, . Assume that

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$e(\phi) := \frac{1}{2} \mid d\phi \mid^2$. The function

$$E_H(\phi) = \int_M [e(\phi) - H(\phi)] d\upsilon_g, \tag{1.1}$$

is called the $H-$energy function of ϕ. Moreover, any critical points of E_H is said to be harmonic map with potential H.

$\mathcal{F}$-harmonic maps as an extension of geodesics , minimal surfaces and harmonic maps were first investigated by Ara in 1999, [1]. Consider a C^2-function $\mathcal{F} : [0, \infty) \longrightarrow [0, \infty)$ such that $F' > 0$ on $(0, \infty)$. The smooth map ϕ is called $\mathcal{F}$*-harmonic* if ϕ is a critical point of the $\mathcal{F}$-energy functional:

$$E_{\mathcal{F}}(\phi) = \int_M \mathcal{F}(\frac{\mid d\phi \mid^2}{2}) d\upsilon_g \tag{1.2}$$

$\mathcal{F}$-energy functional could be categorized as exponential energy, p-energy, or energy when $\mathcal{F}(t)$ is equal to e^t, $(2t)^{\frac{p}{2}}/p \quad (p \geq 4)$ or t, respectively. By calculating the first variation formula for $\mathcal{F}$-energy functional , it can be obtain that

$$\tau_{\mathcal{F}}(\phi) := \mathcal{F}'(\frac{\mid d\phi \mid^2}{2})\tau(\phi) + d\phi(grad_g(\mathcal{F}'(\frac{\mid d\phi \mid^2}{2}))) = 0 \tag{1.3}$$

The operator $\tau_{\mathcal{F}}(\phi)$ is said to be the $\mathcal{F}$*-tension field* of the map ϕ.

In view of physics, $\mathcal{F}$-harmonic maps have a key role in physical cosmology, physics and mechanics. For instance, they are studied to investigate the phenomenon of the quintessence,[3].

In this paper, $\mathcal{F}$-harmonic maps with potential is introduced. Then, the first and second variation formulas for these maps are derived. Finally, the stability of $\mathcal{F}$- harmonic maps with potential into the unit sphere equipped with induced metric is studied.

2. Main results

In this part, first, the notion of $\mathcal{F}-$energy functional with potential H is studied. Then, the variation formulas are obtained. Finally, the stability of these maps are investigated.
Consider the C^3 map $\phi : M \longrightarrow N$ between Riemannian manifolds. Denote the Levi-Civita connection of M, N and $\phi^{-1}TN$ by ${}^M\nabla, {}^N\nabla$ and $\hat{\nabla}$. Let H be a smooth function on N and let $F : [0, \infty) \longrightarrow [0, \infty)$ be a C^3- strictly increasing function . F-bienergy functional with potential H can be considered as follows:

$$E_{\mathcal{F},H}(\phi) = \int_M (\mathcal{F}(\frac{\mid d\phi \mid^2}{2}) + H \circ \phi) d\upsilon_g. \tag{2.1}$$

A map ϕ is said to be $\mathcal{F}$*-harmonic* with potential H if ϕ is a critical point of the F-energy functional. F-harmonic maps with potential H can be categorized as harmonic , p-harmonic or exponentially harmonic when $F(t)$ is equal to t, $(2t)^{\frac{p}{2}}/p$ ($p \geq 4$) or e^t respectively. By choosing a local orthonormal frame field $\{e_i\}$ on M, The $F-H-$tension field of ϕ, $\tau_{F,H}(\phi)$, is defined by

$$\tau_{F,H}(\phi) = F'(\frac{\mid d\phi \mid^2}{2})\tau(\phi) + d\phi(gradF'(\frac{\mid d\phi \mid^2}{2})) +^N \nabla H \circ \phi, \quad (2.2)$$

here $\tau(\phi) = \sum_{i=1}^{m}\{\hat{\nabla}_{e_i} d\phi(e_i) - d\phi(^M\nabla_{e_i} e_i)\}$ is the tension field of ϕ. According to the above notations we get

Lemma 2.1. *(The first variation formula) Let $\phi : (M,g) \longrightarrow (N,h)$ be a smooth map. Then*

$$\frac{d}{dt}E_{F,H}(\phi_t) \mid_{t=0} = -\int_M h(\tau_{F,H}(\phi), V)dv_g, \quad (2.3)$$

where $V = \frac{d\phi_t}{dt} \mid_{t=0}$.

By 2.1, the notion of $F-$harmonic map with potential H for the functional $E_{F,H}$ can be defined as follows

Definition 2.2. A C^2 map ϕ is said to be $F-$harmonic with potential H for the functional $E_{F,H}$ if $\tau_{F,H}(\phi) = 0$.

Definition 2.3. Let $\phi : (M,g) \longrightarrow (N,h)$ be an $F-$harmonic map with potential H, and let $\phi_t : M \longrightarrow N$ $(-\epsilon < t < \epsilon)$ be a smooth variation of $\phi_0 = \phi$ and $V = \dfrac{\partial \phi_t}{\partial t} \mid_{t=0}$. Setting

$$I(V) = \frac{d^2}{dt^2}E_{F,H}(\phi_t) \mid_{t=0}$$

The map ϕ is said to be stable if $I(V) \geq 0$ for any vector field V along ϕ.

By computing the second variation formula, it can be seen that

$$\begin{aligned} I(V) = &\int_M F''(\frac{\mid d\phi \mid^2}{2})\langle\hat{\nabla}V, d\phi\rangle^2 dv_g \\ &+ \int_M F'(\frac{\mid d\phi \mid^2}{2})\Big\{\langle\mid \hat{\nabla}V \mid^2 - h(trace_g\, {}^N R(V, d\phi)d\phi \\ &- (\nabla_V^N grad^N H) \circ \phi, V)\Big\} dv_g \end{aligned} \quad (2.4)$$

where $\mid \hat{\nabla}V \mid$ denotes the Hilbert-Schmidt norm of the $\hat{\nabla}V \in \Gamma(T^*M \times \phi^{-1}TN)$. By (2.4), we have

Theorem 2.4. *Let $\phi : (M, g) \longrightarrow \mathbb{S}^n$ be a stable $F-$ harmonic map with potential H from a Riemannian manifold (M, g) to $\mathbb{S}^n (n > 2)$, and let $\triangle^{\mathbb{S}^n} H \circ \phi \geq 0$. Suppose that $(\mathcal{F}(e(\phi)))'' < 0$ for $n < 2$. Then ϕ is constant.*

According to (2.4), we get

Corollary 2.5. *Let $\phi : (M, g) \longrightarrow \mathbb{S}^n$ be a stable $\mathcal{F}-$harmonic map with potential H from a Riemannian manifold (M, g) to $\mathbb{S}^n (n > 2)$. Suppose that H is an affine function. Then ϕ is constant.*

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